Inequality
https://www.linkedin.com/groups/8313943/8313943-6431528200093990916
Prove that if a, b and c are positive real numbers such that
$abc = 1$, then $a^3 + b^3 + c^3 + \dfrac{(a^2+2)(b^2+2)(c^2+2)}{a^2+b^2+c^2} \geq 12$.

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Noting that $\dfrac{a^3+b^3+c^3}{3} \geq \left(\dfrac{a^2+b^2+c^2}{3}\right)^{\frac{3}{2}} \Leftrightarrow a^3+b^3+c^3 \geq \dfrac{(a^2+b^2+c^2)^{\frac{3}{2}}}{\sqrt{3}}$
we will try to prove stronger inequality

(**1**) $\quad \dfrac{(a^2+b^2+c^2)^{\frac{3}{2}}}{\sqrt{3}} + \dfrac{(a^2+2)(b^2+2)(c^2+2)}{a^2+b^2+c^2} \geq 12.$

Denoting $x := a^2, y := b^2, z := c^2$ we can equivalently rewrite inequality (**1**) as inequality

(**2**) $\quad \dfrac{(x+y+z)^{\frac{3}{2}}}{\sqrt{3}} + \dfrac{(x+2)(y+2)(z+2)}{x+y+z} \geq 12,$ where $x, y, z > 0$ and $xyz = 1.$

Let $s := x+y+z, p := xy+yz+zx$. Then (**2**) becomes

$$\frac{s\sqrt{s}}{\sqrt{3}} + \frac{9+2p+4s}{s} \geq 12 \Leftrightarrow \frac{s\sqrt{s}}{\sqrt{3}} + \frac{9+2p}{s} \geq 8.$$

Since $p \geq \sqrt{3s}$ $\left(p^2 = (xy+yz+zx)^2 \geq 3xyz(x+y+z) = 3s\right)$ then
$\dfrac{s\sqrt{s}}{\sqrt{3}} + \dfrac{9+2p}{s} \geq \dfrac{s\sqrt{s}}{\sqrt{3}} + \dfrac{9+2\sqrt{3s}}{s}$ and, denoting $t := \sqrt{\dfrac{3}{s}}$, we obtain
$\dfrac{s\sqrt{s}}{\sqrt{3}} + \dfrac{9+2\sqrt{3s}}{s} = 3\left(\sqrt{\dfrac{s}{3}}\right)^3 + 3 \cdot \dfrac{3}{s} + 2\sqrt{\dfrac{3}{s}} = \dfrac{3}{t^3} + 3t^2 + 2t.$

Noting that $t \leq 1$ (because $s = x+y+z \geq 3(xyz)^{\frac{1}{3}} = 3$) we obtain
$\dfrac{3}{t^3} + 3t^2 + 2t - 8 - \dfrac{(1-t)(3(1-t^3) + 3t(1-t^2) + 3t^2(1-t^2) + t^3)}{t^3} \geq 0.$