Hard?

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Let $\triangle ABC$, with area Δ. Prove that $\sum \frac{ar_a}{br_b+cr_c} \geq \frac{12\Delta}{16R^2-r^2-4\Delta}$.

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Note that by Cauchy Inequality $\sum \frac{ar_a}{br_b+cr_c} = \sum \frac{(ar_a)^2}{abr_br_b+car_cr_a} \geq \frac{\left(\sum ar_a\right)^2}{2\sum abr_br_b}$

and $\frac{\left(\sum ar_a\right)^2}{2\sum abr_br_b} \geq \frac{3}{2} \Leftrightarrow \left(\sum ar_a\right)^2 \geq 3\left(\sum abr_br_b\right) \Leftrightarrow \sum (ar_a-br_b)^2 \geq 0.$

Hence, $\sum \frac{ar_a}{br_b+cr_c} \geq \frac{3}{2}$.

As the next step we will find $\max\left(\frac{12\Delta}{16R^2-r^2-4\Delta}\right)$.

Let $x := s-a, y := s-b, z := s-c$ and $p := xy+yz+zx, q := xyz.$ Then $x,y,z > 0$ and $x+y+z = s.$ Assuming $s = 1$ (due homogeneity of the expression) we obtain

$a = 1-x, b = 1-y, c = 1-z, r = \Delta = \sqrt{q}, R = \frac{abc}{4\Delta} = \frac{p-q}{4\sqrt{q}}$ and $\frac{12\Delta}{16R^2-r^2-4\Delta} =$

$$\frac{12\sqrt{q}}{16\cdot\left(\frac{p-q}{4\sqrt{q}}\right)^2-\sqrt{q}-4\sqrt{q}} = \frac{12q\sqrt{q}}{(p-q)^2-5q\sqrt{q}}.$$

Since $q \leq \frac{p^2}{3}\left(3xyz(x+y+z) \leq (xy+yz+zx)^2\right), p \leq \frac{1}{3}\left(3(xy+yz+zx) \leq (x+y+z)^2\right)$

and $\frac{12q\sqrt{q}}{(p-q)^2-5q\sqrt{q}}$ is increasing by $q \in (0, p^2/3]$ then

$$\frac{12q\sqrt{q}}{(p-q)^2-5q\sqrt{q}} \leq \frac{12(p^2/3)\sqrt{p^2/3}}{(p-p^2/3)^2-(p^2/3)^2-4(p^2/3)\sqrt{p^2/3}} = \frac{12p}{3\sqrt{3}-2p\sqrt{3}-4p} \leq$$

$$\frac{12\cdot\frac{1}{3}}{3\sqrt{3}-2\cdot\frac{1}{3}\sqrt{3}-4\cdot\frac{1}{3}} = \frac{84\sqrt{3}+48}{131}$$ (because $\frac{12p}{3\sqrt{3}-2p\sqrt{3}-4p}$ increase by $p \in (0,1/3]$).

Noting that $\frac{3}{2} > \frac{84\sqrt{3}+48}{131} \Leftrightarrow \frac{1}{2} > \frac{28\sqrt{3}+16}{131} \Leftrightarrow 99 > 56\sqrt{3} \Leftrightarrow 11\cdot 3\sqrt{3} > 56 \Leftrightarrow 121\cdot 27 > 49\cdot 64 \Leftrightarrow 3267 > 3136$

we finally obtain $\sum \frac{ar_a}{br_b+cr_c} \geq \frac{3}{2} > \frac{84\sqrt{3}+48}{131} \geq \frac{12\Delta}{16R^2-r^2-4\Delta}$.

So, the inequality is not hard, but equality in the inequality of the problem never occurs, although $\frac{3}{2}$ is close to $\frac{84\sqrt{3}+48}{131}$ $\left(\frac{3}{2}-\frac{84\sqrt{3}+48}{131} \approx 2.2960\times 10^{-2}\right)$.