

[https://www.linkedin.com/feed/update/urn:li:activity:7272496381951381506?
utm_source=share&utm_medium=member_desktop](https://www.linkedin.com/feed/update/urn:li:activity:7272496381951381506?utm_source=share&utm_medium=member_desktop)

H-840 Proposed by Arkady Alt, San Jose , California, USA.

Prove that $(n-1)(n+1)(2nF_{n+1} - (n+6)F_n)$ is divisible by 150 for any $n \in \mathbb{N}$.

(F_n is n -th Fibonacci number, $n \in \mathbb{N}$)

Solution.

Let $s_n := \frac{(n-1)(n+1)(2nF_{n+1} - (n+6)F_n)}{150}$, $n \in \mathbb{N} \cup \{0\}$. Then $s_0 = 0, s_1 = 0$.

Note that $s_{n+1} = \frac{n(n+2)(2(n+1)F_{n+2} - (n+7)F_{n+1})}{150} =$
 $\frac{n(n+2)(2(n+1)(F_{n+1} + F_n) - (n+7)F_{n+1})}{150} = \frac{n(n+2)((n-5)F_{n+1} + 2(n+1)F_n)}{150}$

and

$$s_{n-1} = \frac{(n-2)n(2(n-1)F_n - (n+5)F_{n-1})}{150} = \frac{(n-2)n(2(n-1)F_n - (n+5)(F_{n+1} - F_n))}{150} =$$

$$\frac{(n-2)n(3(n+1)F_n - (n+5)F_{n+1})}{150}. \text{ Then}$$

$$s_{n+1} - s_n - s_{n-1} = \frac{n(n+2)((n-5)F_{n+1} + 2(n+1)F_n)}{150} - \frac{(n-1)(n+1)(2nF_{n+1} - (n+6)F_n)}{150} -$$

$$\frac{(n-2)n(3(n+1)F_n - (n+5)F_{n+1})}{150} = \frac{(5n^2 + 3n - 2)F_n - 6nF_{n+1}}{50}.$$

Let $h_n := \frac{(5n^2 + 3n - 2)F_n - 6nF_{n+1}}{50}$, $n \in \mathbb{N} \cup \{0\}$. Then $h_0 = 0, h_1 = 0$.

Note that $h_{n+1} = \frac{(5(n+1)^2 + 3(n+1) - 2)F_{n+1} - 6(n+1)F_{n+2}}{50} =$
 $\frac{(5n^2 + 13n + 6)F_{n+1} - 6(n+1)(F_{n+1} + F_n)}{50} = \frac{n(5n+7)F_{n+1} - 6(n+1)F_n}{50}.$

Then

$$h_{n+2} = \frac{(n+1)(5(n+1) + 7)F_{n+2} - 6(n+2)F_{n+1}}{50} = \frac{(5n^2 + 17n + 12)(F_{n+1} + F_n) - 6(n+2)F_{n+1}}{50} =$$

$$\frac{(5n^2 + 17n + 12)F_n + n(5n+11)F_{n+1}}{50} \text{ and, therefore,}$$

$$h_{n+2} - h_{n+1} - h_n = \frac{(5n^2 + 17n + 12)F_n + n(5n+11)F_{n+1}}{50} - \frac{n(5n+7)F_{n+1} - 6(n+1)F_n}{50} -$$

$$\frac{(5n^2 + 3n - 2)F_n - 6nF_{n+1}}{50} = \frac{nF_{n+1} + 2(n+1)F_n}{5}.$$

Let $g_n := \frac{nF_{n+1} + 2(n+1)F_n}{5}$, $n \in \mathbb{N} \cup \{0\}$. Then $g_0 = 0, g_1 = \frac{F_2 + 4F_1}{5} = 1$ and

we have $g_{n+1} = \frac{(n+1)F_{n+2} + 2(n+2)F_{n+1}}{5} = \frac{(n+1)(F_{n+1} + F_n) + 2(n+2)F_{n+1}}{5} =$
 $\frac{(n+1)F_n + (3n+5)F_{n+1}}{5}$. Also, $g_{n-1} = \frac{(n-1)F_n + 2nF_{n-1}}{5} = \frac{(n-1)F_n + 2n(F_{n+1} - F_n)}{5} =$
 $\frac{2nF_{n+1} - (n+1)F_n}{5}.$

Hence,

$$g_{n+1} - g_n - g_{n-1} = \frac{(n+1)F_n + (3n+5)F_{n+1}}{5} - \frac{nF_{n+1} + 2(n+1)F_n}{5} - \frac{2nF_{n+1} - (n+1)F_n}{5} = F_{n+1}.$$

Since g_n is integer for any $n \in \mathbb{N} \cup \{0\}$ (By Math Induction because $g_0 = 0, g_1 = 1$ and $g_{n+1} = g_n + g_{n-1} + F_{n+1}, n \in \mathbb{N}$) and $h_{n+2} = h_{n+1} + h_n + g_n, n \in \mathbb{N} \cup \{0\}$, where $h_0 = h_1 = 0$,

then for any $n \in \mathbb{N} \cup \{0\}$ h_n is integer as well. Thus, $s_{n+1} = s_n + s_{n-1} + h_n, n \in \mathbb{N} \cup \{0\}$, where $s_0 = 0, s_1 = 0$ and h_n is integer for any $n \in \mathbb{N} \cup \{0\}$. Hence, again by Math Induction, we can conclude that s_n is integer for any $n \in \mathbb{N} \cup \{0\}$.