https://www.linkedin.com/feed/update/urn:li:activity:6540151629071687681

4158. **Proposed by George Apostolopoulos**.

Let m_a, m_b and m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Let R and d_a be, respectively, circumradius and distance from the circumcenter to side a. Then by triangle inequality $|m_a - R| \leq d_a$ and, since $d_a = \sqrt{R^2 - \frac{a^2}{4}}, m_a^2 = \frac{2(b^2 + c^2) - a^2}{4}$ we obtain:

$|m_a - R| \leq \sqrt{R^2 - \frac{a^2}{4}} \Leftrightarrow m_a^2 - 2m_a R + R^2 \leq R^2 - \frac{a^2}{4} \Leftrightarrow 4m_a^2 - 8m_a R + a^2 \leq 0 \Leftrightarrow$
$2(b^2 + c^2) - a^2 - 8m_a R + a^2 \leq 0 \Leftrightarrow b^2 + c^2 \leq 4m_a R \Leftrightarrow$

(**1**) $\frac{b^2 + c^2}{4R} \leq m_a$.

Then $\sum_{cyc} m_a \geq \sum_{cyc} \frac{b^2 + c^2}{4R} = \frac{a^2 + b^2 + c^2}{2R} = \frac{4R^2(\sin^2 A + \sin^2 B + \sin^2 C)}{2R} =$
$2R(\sin^2 A + \sin^2 B + \sin^2 C)$ and since $R \geq 2r$ (Euler's Inequality) we obtain
$\sum_{cyc} m_a \geq 4r(\sin^2 A + \sin^2 B + \sin^2 C)$.

Hence, $\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r$.