1036.**Proposed by George Apostolopoulos**, **Messolonghi**, **Greece**.

Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\left(a^3(a+1)+\ b^3(b+1)+\ c^3(c+1)\right)\cdot\frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)}\geq 48.$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Note that for any positive x holds inequality

(**1**) $\quad \dfrac{x^3+3}{x+1}\geq\dfrac{x+3}{2}$.

Indeed, (**1**)$\Leftrightarrow 2(x^3+3)-(x+1)(x+3)\geq 0 \Leftrightarrow (2x+3)(x-1)^2\geq 0.$

Since $\dfrac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)}\geq\dfrac{(a+3)(b+3)(c+3)}{8}$ remains to prove inequality

(**2**) $\quad \sum_{cyc}a^3(a+1)\cdot\prod_{cyc}\dfrac{a+3}{2}\geq 48.$

By AM-GM we have $\sum_{cyc}a^3(a+1)=\sum_{cyc}a^4+\sum_{cyc}a^3\geq 3\sqrt[3]{a^4b^4c^4}+3\sqrt[3]{a^3b^3c^3}=6$

and $\prod_{cyc}\dfrac{a+3}{2}=\dfrac{1}{8}\prod_{cyc}(a+3)\geq\dfrac{1}{8}\prod_{cyc}4\sqrt[4]{a\cdot 1\cdot 1\cdot 1}=8\sqrt[4]{abc}=8.$

Hence, $\sum_{cyc}a^3(a+1)\cdot\prod_{cyc}\dfrac{a+3}{2}\geq 6\cdot 8=48.$

1037.**Proposed by George Apostolopoulos**, **Messolonghi**, **Greece**.

Let P be a point inside the triangle ABC and let D,E,F be the projections of P on the sides BC,CA, and AB, respectively. Prove that

$$\frac{PA+PB+PC}{(EF\cdot FD\cdot DE)^{1/3}}\geq 2\sqrt{3}$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Let $R_a:=PA, R_b:=PB, R_c:=PC$ and $a_p:=EF, b_p:=FD, c_p:=DE$

(that is a_p,b_p,c_p are sidelengths of pedal triangle of point P). Then original inequality in the new notation becomes

(**1**) $\quad \dfrac{R_a+R_b+R_c}{(a_pb_pc_p)^{1/3}}\geq 2\sqrt{3}$

Since quadrilateral $FAEP$ is cyclic (because $PF\perp AB$ and $PE\perp AC$)

and R_a is diameter of circumcircle of quadrilateral $FAEP$

then $\dfrac{a_p}{R_a}=\sin A=\dfrac{a}{2R}$ and, similarly, $\dfrac{b_p}{R_b}=\sin B=\dfrac{b}{2R},\dfrac{c_p}{R_c}=\sin C=\dfrac{c}{2R}$ and inequality

(**1**)

can be rewritten as $\dfrac{R_a+R_b+R_c}{\left(\dfrac{aR_a}{2R}\cdot\dfrac{bR_b}{2R}\cdot\dfrac{cR_c}{2R}\right)^{1/3}}\geq 2\sqrt{3}\Leftrightarrow\dfrac{2R(R_a+R_b+R_c)}{(aR_a\cdot bR_b\cdot cR_c)^{1/3}}\geq 2\sqrt{3}\Leftrightarrow$

$\dfrac{R(R_a+R_b+R_c)}{\sqrt[3]{R_aR_bR_c}}\geq\sqrt{3}\sqrt[3]{abc}\Leftrightarrow\sum_{cyc}\sqrt[3]{\dfrac{R_a^2}{R_bR_c}}\geq\dfrac{\sqrt{3}}{R}\sqrt[3]{abc}$ or $\sum_{cyc}\sqrt[3]{\dfrac{R_a^2}{R_bR_c}}\geq\sqrt{3}\sqrt[3]{\dfrac{sr}{R^2}}$.

Or, inequality (**1**) can be rewritten as $\dfrac{R_a+R_b+R_c}{(R_a\sin A\cdot R_b\sin B\cdot R_c\sin C)^{1/3}}\geq 2\sqrt{3}\Leftrightarrow$

$\dfrac{R_a+R_b+R_c}{(R_aR_bR_c)^{1/3}}\geq 2\sqrt{3}\,(\sin A\sin B\cdot\sin C)^{1/3}.$

Since $\frac{R_a + R_b + R_c}{(R_a R_b R_c)^{1/3}} \geq 3$ suffice to prove that $3 \geq 2\sqrt{3}\,(\sin A \sin B \cdot \sin C)^{1/3} \Leftrightarrow$ $\frac{\sqrt{3}}{2} \geq (\sin A \sin B \cdot \sin C)^{1/3}$.

We have $\frac{\sin A + \sin B + \sin C}{3} \leq \frac{\sqrt{3}}{2}$ (because for $\sin x$ which is concave down on $[0, \pi]$ by Jensen's Inequality holds $\frac{\sin A + \sin B + \sin C}{3} \leq \sin \frac{A + B + C}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$)

and by AM-GM $(\sin A \sin B \cdot \sin C)^{1/3} \leq \frac{\sin A + \sin B + \sin C}{3}$.

(Another way to prove inequality $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

First note that for any $x, y \in [0, \pi]$ holds inequality $\sin x + \sin y \leq 2\sin \frac{x + y}{2}$.

Indeed, $\sin x + \sin y = 2\sin \frac{x + y}{2} \cos \frac{x - y}{2} \leq 2\sin \frac{x + y}{2}$ because $\frac{x + y}{2} \in [0, \pi]$ and $\frac{x - y}{2} \in [-\pi/2, \pi/2]$.

Using inequality $\sin x + \sin y \leq 2\sin \frac{x + y}{2}$ we obtain

$$\sin A + \sin B + \sin C + \sin \frac{\pi}{3} \leq 2\sin \frac{A + B}{2} + 2\sin \frac{C + \frac{\pi}{3}}{2} \leq 4\sin \frac{\frac{A + B}{2} + \frac{C + \frac{\pi}{3}}{2}}{2} =$$

$$4\sin \frac{\pi + \frac{\pi}{3}}{4} = 4 \cdot \sin \frac{\pi}{3} = 2\sqrt{3} \Rightarrow \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}).$$

1038.Proposed by D. M. Bătinețụ-Giurgiu, Matei Basarab National College, Bucharest,
Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Let m be a nonnegative real number and x, y be positive real numbers. Prove that, for any triangle ABC with side lengths a, b, c where $[ABC]$ denotes the area of triangle,

$$\frac{a^{m+2}}{(xb + yc)^m} + \frac{b^{m+2}}{(xc + ya)^m} + \frac{c^{m+2}}{(xa + yb)^m} \geq \frac{4\sqrt{3}}{(x + y)^m} \cdot [ABC].$$

Solution by Arkady Alt , San Jose ,California, USA.

Let $u := \frac{xb + yc}{x + y}, v := \frac{xc + ya}{x + y}, w := \frac{xa + yb}{x + y}$ and $I_m := \frac{a^{m+2}}{u^m} + \frac{b^{m+2}}{v^m} + \frac{c^{m+2}}{w^m}$

then $\sum_{cyc} \frac{a^{m+2}}{(xb + yc)^m} \geq \frac{4\sqrt{3}}{(x + y)^m} \cdot [ABC] \Leftrightarrow I_m \geq 4\sqrt{3} \cdot [ABC]..$

We will prove that $I_{m+1} \geq I_m$ for any $m \in \mathbb{N} \cup \{0\}$.

Noting that $I_0 = a^2 + b^2 + c^2$ and using inequality $\frac{\alpha^2}{\beta} \geq 2\alpha - \beta, \alpha, \beta > 0$ we obtain

$$I_1 = \sum_{cyc} \frac{a^3}{u} = \sum_{cyc} a \cdot \frac{a^2}{u} \geq \sum_{cyc} a(2a - u) = I_0 + \sum_{cyc} a(a - u) =$$

$$I_0 + \sum_{cyc} \left(a^2 - \frac{a(xb + yc)}{x + y} \right) = I_0 + a^2 + b^2 + c^2 - \sum_{cyc} \frac{a(xb + yc)}{x + y} =$$

$$I_0 + (a^2 + b^2 + c^2 - ab - bc - ca) \geq I_0.$$

Taking inequality $I_1 \geq I_0$ as base of Math Induction and assuming for any $m \in \mathbb{N}$

that $I_m \geq I_{m-1}$ we will prove that $I_{m+1} \geq I_m$.

We have $I_{m+1} = \sum_{cyc} \frac{a^{m+3}}{u^{m+1}} = \sum_{cyc} \frac{a^{m+1}}{u^m} \cdot \frac{a^2}{u} \geq \sum_{cyc} \frac{a^{m+1}}{u^m}(2a-u) = I_m + \sum_{cyc}\left(\frac{a^{m+2}}{u^m} - \frac{a^{m+1}}{u^{m-1}}\right) =$
$I_m + (I_m - I_{m-1}) \geq I_m$.■

Since $(I_m)_{m\geq 0}$ is increasing sequence then $I_m \geq I_0 = a^2+b^2+c^2$ and

(**1**) $\quad a^2+b^2+c^2 \geq 4\sqrt{3} \cdot [ABC]$ (Weitzenböck's inequality)

we obtain $I_m \geq 4\sqrt{3} \cdot [ABC]$.

(Or, direct proof of inequality $a^2+b^2+c^2 \geq 4\sqrt{3} \cdot [ABC]$:

Let $x := s-a, y := s-b, z := s-c$ where s is semiperimeter and let $p := xy+yz+zx$, $q := xyz$. Also, assume (due to homogeneity) that $s := 1$. Then $a^2+b^2+c^2 = 2(1-p)$, $[ABC] = \sqrt{q}$ and inequality (**1**) become $1-p \geq 2\sqrt{3} \cdot \sqrt{q}$.

Since $p^2 = (xy+yz+zx)^2 \geq 3xyz(x+y+z) = 3q$ and

$1 = (x+y+z)^2 \geq 3(xy+yz+zx) = p$

we obtain $1-p-2\sqrt{3} \cdot \sqrt{q} = 1-3p+2\left(p-\sqrt{3q}\right) \geq 0$).