

**Variations on theme of Walker's Inequality.**

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The following several problems represent inequalities between which at first glance there is nothing common but in fact all of them are equivalent to Wolker's Inequality which represent very convenient and useful quadratic  $R, r$  minorant for square of semiperimeter  $s$  of an acute triangle.

(Although the problems listed below are borrowed from different sources, however, all their solutions presented below are solutions of the author of this notes).

**Problem 1.**

Prove that in an acute triangle with sidelengths  $a, b, c$ , circumradius  $R$  and inradius  $r$  holds inequality

$$(1) \quad a^2 + b^2 + c^2 \geq 4(R + r)^2. \quad [1]$$

**Solution.**

$$\text{We have } a^2 + b^2 + c^2 \geq 4(R + r)^2 \iff 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \geq 4R^2\left(1 + \frac{r}{R}\right)^2 \iff$$

$$\sin^2 A + \sin^2 B + \sin^2 C \geq (\cos A + \cos B + \cos C)^2 \iff \sum_{cyc} (1 - \cos^2 A) \geq \sum_{cyc} \cos^2 A + 2 \sum_{cyc} \cos B \cos C \iff \sum_{cyc} (\cos A + \cos B)^2 \leq 3.$$

Since by Cauchy Inequality

$$(\cos A + \cos B)^2 \leq (a \cos B + b \cos A) \left( \frac{\cos B}{a} + \frac{\cos A}{b} \right) = c \left( \frac{\cos B}{a} + \frac{\cos A}{b} \right)$$

$$\begin{aligned} \text{Then } \sum_{cyc} (\cos A + \cos B)^2 &\leq \sum_{cyc} \left( \frac{c \cos B}{a} + \frac{c \cos A}{b} \right) = \\ &= \frac{c \cos B}{a} + \frac{c \cos A}{b} + \frac{a \cos C}{b} + \frac{a \cos B}{c} + \frac{b \cos A}{c} + \frac{b \cos C}{a} = \\ &= \sum_{cyc} \left( \frac{c \cos B + b \cos C}{a} \right) = \sum_{cyc} \frac{a}{a} = 3. \end{aligned}$$

**Remark1.** On the way of the solution was obtained inequality

$$(2) \quad \sum_{cyc} (\cos A + \cos B)^2 \leq 3. \quad [2]$$

which is trigonometric equivalent of inequality (1) and holds in any acute-angled triangle  $ABC$ .

**Remark 2.**

Since  $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$  then

$$a^2 + b^2 + c^2 \geq 4(R + r)^2 \iff 2(s^2 - 4Rr - r^2) \geq 4(R + r)^2 \iff$$

$$(W) \quad s^2 \geq 2R^2 + 8Rr + 3r^2$$

where (W) is **Walker's Inequality** for an acute triangles).

**Problem 2**

Prove that in any triangle the following inequality holds

$$(3) \quad \left( \sin \frac{A}{2} + \sin \frac{B}{2} \right)^2 + \left( \sin \frac{B}{2} + \sin \frac{C}{2} \right)^2 + \left( \sin \frac{C}{2} + \sin \frac{A}{2} \right)^2 \leq 3.$$

**Solution.**

Since sine of half argument not very comfortable object in considerations of the symmetric functions from  $\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}$  then, using cosine-theorem we obtain metrical representation for  $\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}$  and metrical equivalent of original inequality.

$$\text{Namely, } c^2 = a^2 + b^2 - 2ab \cos C = (a-b)^2 + 4ab \sin^2 \frac{C}{2} \implies$$

$$\sin^2 \frac{C}{2} = \frac{(s-a)(s-b)}{ab}$$

$$\text{and similarly we get } \sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}, \sin^2 \frac{B}{2} = \frac{(s-a)(s-c)}{ac}.$$

Then by Cauchy Inequality

$$\begin{aligned} \left( \sin \frac{A}{2} + \sin \frac{B}{2} \right)^2 &= \frac{s-c}{c} \left( \sqrt{s-b} \cdot \frac{1}{\sqrt{b}} + \sqrt{s-a} \cdot \frac{1}{\sqrt{a}} \right)^2 \leq \\ \frac{s-c}{c} \cdot \left( \frac{1}{a} + \frac{1}{b} \right) (s-a+s-b) &= \frac{(a+b-c)(a+b)c}{2abc} = \\ \frac{(a^2+b^2)c - c^2(a+b) + 2abc}{2abc} &= 1 + \frac{(a^2+b^2)c - c^2(a+b)}{2abc} \end{aligned}$$

and, therefore,

$$\begin{aligned} \sum_{cyc} \left( \sin \frac{A}{2} + \sin \frac{B}{2} \right)^2 &\leq 3 + \sum_{cyclic} \frac{(a^2+b^2)c - c^2(a+b)}{2abc} = \\ 3 + \frac{1}{2abc} \left( \sum_{cyc} (a^2+b^2)c - \sum_{cyc} c^2(a+b) \right) &= 3. \end{aligned}$$

**Remark 1.**

$$\sum_{cyc} \left( \sin \frac{A}{2} + \sin \frac{B}{2} \right)^2 \leq 3 \iff \left( \sum_{cyc} \sin \frac{A}{2} \right)^2 \leq \sum_{cyc} \left( 1 - \sin^2 \frac{A}{2} \right) \iff$$

$$(4) \quad \left( \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \right)^2 \leq \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \quad [3]$$

**Remark 2.**

$$\sum_{cyc} \left( \sin \frac{A}{2} + \sin \frac{B}{2} \right)^2 \leq 3 \iff 2 \sum_{cyc} \sin^2 \frac{A}{2} + 2 \sum_{cyc} \sin \frac{A}{2} \sin \frac{B}{2} \leq 3 \iff$$

$$\sum_{cyc} \sin \frac{A}{2} \sin \frac{B}{2} \leq \sum_{cyc} \left( 1 - 2 \sin^2 \frac{A}{2} \right) \iff 2 \sum_{cyc} \sin \frac{A}{2} \sin \frac{B}{2} \leq \sum_{cyc} \cos A \iff$$

$$(5) \quad \sum_{cyc} \sin \frac{A}{2} \sin \frac{B}{2} \leq \frac{1}{2} \left( 1 + \frac{r}{R} \right).$$

**Remark 3.**

$$\text{Let } \alpha := \frac{\pi - A}{2}, \beta := \frac{\pi - B}{2}, \gamma := \frac{\pi - C}{2}.$$

Since  $\alpha, \beta, \gamma \in (0, \pi/2) > 0, \alpha + \beta + \gamma = \pi$

and, therefore, can be considered as angles of an acute triangle with side-lengths  $a, b, c$ , circumradius  $R$  and inradius  $r$  then inequality (4) becomes

$$\left( \sum_{cyc} \cos \left( \frac{\pi}{2} - \frac{A}{2} \right) \right)^2 \leq 2 \sum_{cyc} \sin^2 \left( \frac{\pi}{2} - \frac{A}{2} \right) \iff \left( \sum_{cyc} \cos \alpha \right)^2 \leq 2 \sum_{cyc} \sin^2 \alpha \iff$$

$$\left(1 + \frac{r}{R}\right)^2 \leq 2 \sum_{cyc} \frac{a^2}{4R^2} \iff (1).$$

**Problem 3.**

Let real  $x, y, z > 0$  be any such that  $x^2 + y^2 + z^2 + 2xyz = 1$ . Prove that

$$(6) \quad x^2 + y^2 + z^2 + xy + yz + zx \leq 3/2.$$

**Solution.**

$$\text{We have } x^2 + y^2 + z^2 + xy + yz + zx \leq 3/2 \iff \sum_{cyc} (x + y)^2 \leq 3,$$

Since any positive solution  $(x, y, z)$  of equation  $x^2 + y^2 + z^2 + 2xyz = 1$  can be represented as

$$(x, y, z) = \left( \sqrt{\frac{bc}{(a+b)(c+a)}}, \sqrt{\frac{ca}{(b+c)(a+b)}}, \sqrt{\frac{ab}{(c+a)(b+c)}} \right), \forall (a, b, c > 0)$$

then, using Cauchy Inequality, we obtain

$$\begin{aligned} \sum_{cyc} (x + y)^2 &= \sum_{cyc} \left( \sqrt{\frac{bc}{(a+b)(c+a)}} + \sqrt{\frac{ca}{(b+c)(a+b)}} \right)^2 = \\ &= \sum_{cyc} \frac{c}{a+b} \left( \sqrt{\frac{b}{c+a}} + \sqrt{\frac{a}{b+c}} \right)^2 = \sum_{cyc} \frac{c}{a+b} \left( \sqrt{b} \cdot \frac{1}{\sqrt{c+a}} + \sqrt{a} \cdot \frac{1}{\sqrt{b+c}} \right)^2 \leq \\ &= \sum_{cyc} \frac{c}{a+b} (b+a) \cdot \left( \frac{1}{c+a} + \frac{1}{b+c} \right) = \sum_{cyc} \left( \frac{c}{c+a} + \frac{c}{b+c} \right) = 3. \end{aligned}$$

**Remark.**

Taking in account that  $\{(x, y, z) \mid x, y, z > 0 \text{ and } x^2 + y^2 + z^2 + 2xyz = 1\} = \{(\cos \alpha, \cos \beta, \cos \gamma) \mid x, y, z > 0 \text{ and } \alpha, \beta, \gamma \in (0, \pi/2), \alpha + \beta + \gamma = \pi\}$

we can rewrite inequality (6) as

$$\sum_{cyc} \cos^2 \alpha + \sum_{cyc} \cos \alpha \cos \beta \leq 3/2 \iff 2 \sum_{cyc} \cos^2 \alpha + 2 \sum_{cyc} \cos \alpha \cos \beta \leq 3 \iff$$

(2).

(here numbers  $\alpha, \beta, \gamma$  are interpreted as angles of an acute triangle).

Thus, up to notations and interpretations here is the chain of equivalent inequalities

(W)  $\iff$  (1)  $\iff$  (2)  $\iff$  (3)  $\iff$  (4)  $\iff$  (5)  $\iff$  (6) and proof of any of them at the same time

is proof for all others.

1. Zadachnik Kvanta, Problem M1178, p.153, Proposed by Z.A.Skopets.
2. Problems from the Book, 2008, XYZ Press, Problem 12,p.22..
3. Problems from the Book, 2008, XYZ Press, Problem 9, p.21,  
Proposed by Titu Andreescu, USAMO 2011.