Notes about Ellipse
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0.1 Definition of Ellipse and its equation.

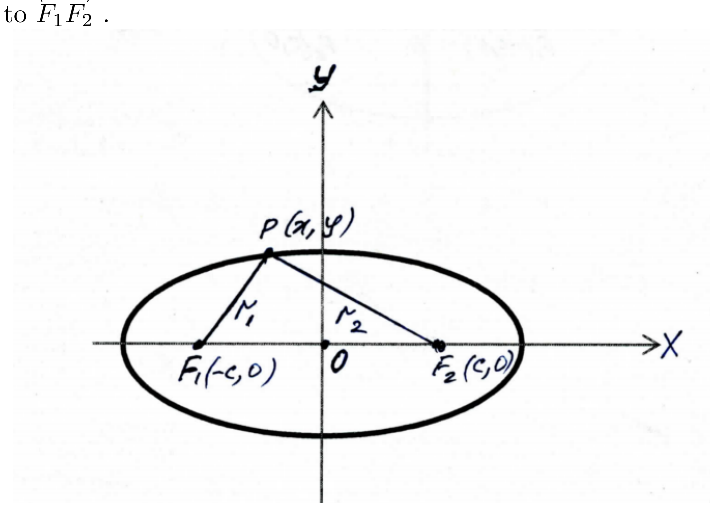
For two given points F_1, F_2 on plane π and given positive real number a we define Ellipse as set

$$\mathcal{E}\ (F_1, F_2; a) := \{M \mid M \in \pi \text{ and } F_1M + F_2M = 2a\}$$

Let $c := \dfrac{F_1F_2}{2}$.Since $F_1M + F_2M \geq F_1F_2$ we should claim $a \geq c$ because, otherwise, $\mathcal{E}\ (F_1, F_2; a) = \varnothing$.

Since in case $a = c$ we have $\mathcal{E}\ (F_1, F_2; a) = \overline{F_1F_2}$ then further we assume $a > c$.

Consider rectangular system of coordinates with origin in point O which bisect segment $\overline{F_1F_2}$ and with line $\overleftrightarrow{F_1F_2}$ as axis OX and perpendicular line in O to $\overleftrightarrow{F_1F_2}$.



Then in such chosen coordinate system we have $F_1(-c, 0)$, $F_2(c, 0)$, $M(x, y)$ and representation of $\mathcal{E}\ (F_1, F_2; a)$ in form of equation is

(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,where $b := \sqrt{a^2 - c^2}$.

(F_1M and F_2M we call focal radii).

Let $r_1 = F_1M = \sqrt{(x+c)^2 + y^2}$ and $r_2 = F_2M = \sqrt{(x-c)^2 + y^2}$ then $r_1 + r_2 = 2a$ and since $2\left(c^2 + x^2 + y^2\right) = r_1^2 + r_2^2 < (r_1 + r_2)^2 = 4a^2 \implies c^2 + x^2 + y^2 < 2a^2$ we have $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \iff (x+c)^2 + y^2 + (x-c)^2 + y^2 + 2\sqrt{(x+c)^2 + y^2} \cdot \sqrt{(x-c)^2 + y^2} = 4a^2 \iff x^2 + y^2 + c^2 + \sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2} = 2a^2 \iff$

$\left(x^2+y^2+c^2\right)^2-4c^2x^2=\left(2a^2-\left(c^2+x^2+y^2\right)\right)^2 \iff -4c^2x^2=4a^4-4a^2\left(c^2+x^2+y^2\right) \iff$
$\iff a^2\left(x^2+y^2\right)-c^2x^2=a^4-a^2c^2 \iff b^2x^2+a^2y^2=a^2b^2 \iff$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Now we will prove *sufficiency, that is:*

If (x,y) *satisfies* ***(1)*** *then* $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$

Let x,y satisfies to equation **(1)**. Then $|x|\le a,\ |y|\le b$ and since $y^2=b^2\left(1-\dfrac{x^2}{a^2}\right)$ we obtain $r_1:=PF_1=\sqrt{(x+c)^2+y^2}=\sqrt{x^2+2cx+c^2+b^2-\dfrac{b^2x^2}{a^2}}=\sqrt{x^2\left(1-\dfrac{b^2}{a^2}\right)+2cx+a^2}=\sqrt{\dfrac{x^2c^2}{a^2}+2cx+a^2}=\sqrt{\left(a+\dfrac{xc}{a}\right)^2}=\left|a+\dfrac{cx}{a}\right|$ and, similarly, $r_2:=PF_2=\sqrt{(x-c)^2+y^2}=\left|a-\dfrac{cx}{a}\right|$.

Thus, $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=\left|a+\dfrac{cx}{a}\right|+\left|a-\dfrac{cx}{a}\right|$. Since $|x|\le a$ then $c\,|x|\le ca<a^2 \iff \left|\dfrac{cx}{a}\right|<a$ and, therefore, $\left|a+\dfrac{cx}{a}\right|+\left|a-\dfrac{cx}{a}\right|=a+\dfrac{cx}{a}+a-\dfrac{cx}{a}=2a.$

By the way we obtain lengths of the focal radii $r_1=a+\dfrac{cx}{a}$ and $r_2=a-\dfrac{cx}{a}$.

Let $p:=\dfrac{a^2}{c}$ and $e:=\dfrac{c}{a}<1$.Number e we call eccentricity of the ellipse and lines $x=-p, x=p$ have name of left and right directrix of ellipse,respectively. Then $r_1=\dfrac{c}{a}\left(\dfrac{a^2}{c}+x\right)=e\,(p+x) \iff \dfrac{r_1}{p+x}=e,\ r_2=\dfrac{c}{a}\left(\dfrac{a^2}{c}-x\right)=e\,(p-x) \iff \dfrac{r_2}{p-x}=e$

that is ratio of left focal radius to distance between point $P\,(x,y)$ on the ellipse to the left directrix $x=-p$ is equal to ratio of right focal radius to distance between point $P\,(x,y)$ on the ellipse to the right directrix $x=p$ and equal to eccentricity of the ellipse.

Remark.

We will prove that Ellipse can be defined by another geometric property, namely if point $M\,(x,y)$ belong to ellipse $\mathcal{E}\,(F_1,F_2;a)$ then there are two lines $l_1,l_2\perp\overleftrightarrow{F_1F_2}$ for which $\dfrac{F_iM}{dist\,(M,l_i)}=\dfrac{c}{a}>1, i=1,2.$

Each such lines we call directrix and number $e:=\dfrac{c}{a}$ we call eccentricities.

Let $\mathcal{E}\,(F_1,F_2;a)$ be ellipse on plane XOY. Then for any $M\,(x,y)$ such that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and line $l_2:x=p>0$ we have $F_2M=\sqrt{(x-c)^2+y^2}, dist\,(M,l_2)=p-x$ (here p is undetermined) and, therefore,

$\dfrac{F_iM}{dist\,(M,l_2)}=e \iff \sqrt{(x-c)^2+y^2}=e\,(p-x) \iff (x-c)^2+y^2=$

$e^2(p-x)^2 \iff x^2-2cx+c^2+y^2=e^2p^2-2e^2px+e^2x^2 \iff x^2(1-e^2)-2x(c-e^2p)+c^2-e^2p^2+y^2=0.$

Since $x^2(1-e^2)+y^2+c^2=x^2\left(1-\dfrac{c^2}{a^2}\right)+y^2+c^2=x^2\cdot\dfrac{b^2}{a^2}+y^2+c^2=b^2+c^2=a^2$ then $\sqrt{(x-c)^2+y^2}=e(p-x) \iff a^2-e^2p^2-2x(c-e^2p)=0$ for any $|x|\leq a$ and in particular for $x=0$.Hence, $p=\dfrac{a}{e}=\dfrac{a^2}{c} \iff c-e^2p=0.$

Thus, $p=\dfrac{a^2}{c}$ and, therefore, $d:=dist(M,l_2)=p-c=\dfrac{b^2}{c}$.So, equation of l_2 is $x=\dfrac{b^2}{c}$.

It is right directrix. Respectively left directrix is $x=-\dfrac{b^2}{c}$.

0.2 To Graphing of ellipse.

Due to symmetry of ellipse $\mathcal{E}=\left\{(x,y)\mid x,y\in\mathbb{R} \text{ and } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\right\}$ with respect to axes OX,OY suffices to consider graph of $\mathcal{E}$ only for $x,y\geq 0$.

For such x,y we have $y=f(x):=b\sqrt{1-\dfrac{x^2}{a^2}}$. Obvious that $f(x)$ decrease on $[0,a]$.

Remains to prove that $f(x)$ is concave down on $[0,a]$, that is to prove inequality $f\left(\dfrac{at_1+at_2}{2}\right)\geq\dfrac{f(at_1)+f(at_2)}{2} \iff \sqrt{1-\left(\dfrac{t_1+t_2}{2}\right)^2}\geq\dfrac{1}{2}\left(\sqrt{1-t_1^2}+\sqrt{1-t_2^2}\right) \iff$
$1-\left(\dfrac{t_1+t_2}{2}\right)^2\geq\dfrac{1}{4}\left(1-t_1^2+1-t_2^2+2\sqrt{1-t_1^2}\cdot\sqrt{1-t_2^2}\right) \iff$

$4-t_1^2-t_2^2-2t_1t_2\geq 2-t_1^2-t_2^2+2\sqrt{1-t_1^2}\cdot\sqrt{1-t_2^2} \iff 2-2t_1t_2\geq 2\sqrt{1-t_1^2}\cdot\sqrt{1-t_2^2} \iff 1-t_1t_2\geq\sqrt{1-t_1^2}\cdot\sqrt{1-t_2^2} \iff (t_1-t_2)^2\geq 0.$

0.3 Tangent to Ellipse.

For ellipse (as for the circle) we can use the simplest definition of tangent, namely,

tangent to the ellipse $\mathcal{E}$ to the given point $P(x_0,y_0)\in\mathcal{E}$ is a line which pass through

P and have no more common points with the ellipse.

Let $P(x_0,y_0)$ be the point one the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, that is $\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1.$

Then equation of the ellipse is $\dfrac{x^2-x_0^2}{a^2}+\dfrac{y^2-y_0^2}{b^2}=0.$

Follow to definition of tangent line we will find its equation in the form

$p(x-x_0)+q(y-y_0)=0$ (that is we will find unknown parameters p and q up to

collinearity of vector (p,q)) by claiming (follow to definition of tangent line) uniqueness of solution of the system with respect (x,y).

$$\begin{cases} p(x-x_0)+q(y-y_0)=0 \\ \dfrac{x^2-x_0^2}{a^2}+\dfrac{y^2-y_0^2}{b^2}=0 \end{cases}$$

Since $p(x-x_0)+q(y-y_0)=0 \iff \exists(t\in\mathbb{R})\,[x-x_0=qt,y-y_0=-pt]$ then by substitution

$(x,y)=(x_0+qt,y_0-pt),t\in\mathbb{R}$ in equation $\dfrac{x^2-x_0^2}{a^2}+\dfrac{y^2-y_0^2}{b^2}=0$ we obtain

$\dfrac{qt(2x_0+qt)}{a^2}+\dfrac{(-pt)(2y_0-pt)}{b^2}=0 \iff t\left(\left(\dfrac{q^2}{a^2}+\dfrac{p^2}{b^2}\right)t+\dfrac{2qx_0}{a^2}-\dfrac{2py_0}{b^2}\right)=0.$

The latter equation has only solution iff $\dfrac{2qx_0}{a^2}-\dfrac{2py_0}{b^2}=0 \iff (p,q)=k\left(b^2x_0,a^2y_0\right),k\in\mathbb{R}\diagdown\{0\}$

Thus, we obtain the following equation of tangent line

$b^2x_0(x-x_0)+a^2y_0(y-y_0)=0 \iff b^2xx_0-b^2x_0^2+a^2yy_0-a^2y_0^2=0 \iff \dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2} \iff$

(T) $\quad \dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1.$

(Equation of line which tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at point $P(x_0,y_0)$ we can

also find in the parametric form

(!) $\quad \begin{cases} x=x_0+pt \\ y=y_0+qt \end{cases}.$

Then system of equation $\begin{cases} x=x_0+pt \\ y=y_0+qt \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \end{cases}$ must have only one solution, namely such t

for which corespondent (x,y) equal (x_0,y_0), that is $t=0$.

We have $\dfrac{(x_0+pt)^2}{a^2}+\dfrac{(y_0+qt)^2}{b^2}=1 \iff$

$\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+t^2\left(\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}\right)+2t\left(\dfrac{px_0}{a^2}+\dfrac{qty_0}{b^2}\right)=1 \iff$

$t\left(t\left(\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}\right)+2\left(\dfrac{px_0}{a^2}+\dfrac{qy_0}{b^2}\right)\right)=0.$

The latter equation has only one root iff $\dfrac{px_0}{a^2}+\dfrac{qy_0}{b^2}=0 \iff (p,q)=k\left(a^2y_0,-b^2x_0\right)$.

Then (!) $\iff \dfrac{x-x_0}{p}=\dfrac{y-y_0}{q} \iff \dfrac{x-x_0}{a^2y_0}=\dfrac{y-y_0}{-b^2x_0} \iff \dfrac{x-x_0}{a^2y_0}+$

$\frac{y-y_0}{b^2 x_0} = 0 \iff$
$\frac{(x-x_0)\,x_0}{a^2} + \frac{(y-y_0)\,y_0}{b^2} = 0 \iff \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$

(Remark. Multiplying $x = x_0 + pt$ by $\frac{x_0}{a^2}$,and $y = y_0 + qt$ by $\frac{y_0}{b^2}$ and adding after,

we obtain $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + t\left(\frac{px_0}{a^2} + \frac{qy_0}{b^2}\right) = 1$, because $\frac{px_0}{a^2} + \frac{qy_0}{b^2} = 0$ and $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$).

(*Here is another, a slightly different way of deriving the equation of*

Tangent line to ellipse.

Let the ellipse $\mathcal{E}$ be given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let $M(x_0, y_0)$ be a point on the ellipse.

Ellipse as a closed curve divides the plane into two parts - the outer and internal with respect to itself, namely the point $P(x, y)$ on the plane is called external with respect to the ellipse if $\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1$ and internal if $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$.

The boundary of the inner and outer region is the ellipse itself, that is, the points of the plane whose coordinates (x, y) are subject to the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Definition.

The line $mx + ny = l$ passing through the point $M(x_0, y_0)$ which lying on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called the tangent to the ellipse at the point M if the line does not contain any points of the plane interior with respect to the ellipse, that is, for any (x, y) such that $mx + ny = l$ must holds inequality $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$.

The fact that a point $M(x_0, y_0)$ belongs to a line $mx + ny = l$ and an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ means that $mx_0 + ny_0 = l$ and $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

Taking these equations into account, the equation of the line and the inequality take the form $m(x - x_0) + n(y - y_0) = 0$ and $\frac{x^2 - x_0^2}{a^2} + \frac{y^2 - y_0^2}{b^2} \geq 0$, respectively.

Since that at least one of two numbers m, n is different from zero, let it be n, then denoting $\frac{x - x_0}{n}$ via t and, by substitution $x - x_0 = nt$ in to equation $m(x - x_0) + n(y - y_0) = 0$, we obtain $mnt + n(y - y_0) = 0 \iff y - y_0 = -mt$.

On the other hand, for every $t \in \mathbb{R}$, numbers $x = x_0 + nt$ and $y = y_0 - mt$ satisfies to equation $m(x - x_0) + n(y - y_0) = 0$.

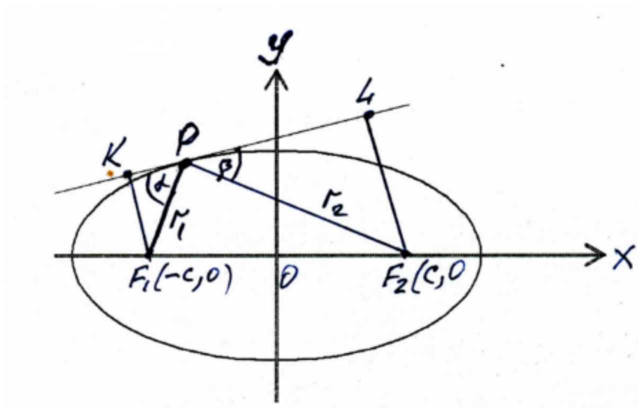
Thus formulas $x = x_0 + nt$ and $y = y_0 - mt$, where $t \in \mathbb{R}$, gives a parametric representation of all solutions of equation $m(x - x_0) + n(y - y_0) = 0$,using which we can rewrite inequality $\frac{x^2 - x_0^2}{a^2} + \frac{y^2 - y_0^2}{b^2} \geq 0$ in the following way:

$\dfrac{nt\left(2x_0+nt\right)}{a^2}-\dfrac{mt\left(2y_0-mt\right)}{b^2}\geq 0 \iff t\left(t\left(\dfrac{n^2}{a^2}+\dfrac{m^2}{b^2}\right)+2\left(\dfrac{nx_0}{a^2}-\dfrac{my_0}{b^2}\right)\geq 0\right), t\in \mathbb{R}$.

Since the last inequality holds for all real values of t iff $\dfrac{nx_0}{a^2}-\dfrac{my_0}{b^2}=0 \iff \dfrac{nx_0}{a^2}=\dfrac{my_0}{b^2} \iff n=\dfrac{ky_0}{b^2}$and $m=\dfrac{kx_0}{a^2}, k\in\mathbb{R}\backslash\{0\}$, then equation of tangent line will be $\dfrac{kx_0}{a^2}\left(x-x_0\right)+\dfrac{ky_0}{b^2}\left(y-y_0\right)=0 \iff$

$$\frac{x_0x}{a^2}-\frac{x_0^2}{a^2}+\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=0 \iff \frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1).$$

0.3.1 Reflection property of tangent to ellipse.



Let line l is tangent to ellipse $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. We will prove that focal radii to the point of tangency form equal angles with the line l (that is $\alpha=\beta$ on pic). Let F_1K and F_2L be perpendiculars to the line that tangent to the ellipse at point $P\left(x_0,y_0\right)$.

We have $r_1=a+\dfrac{x_0c}{a}=\dfrac{a^2+x_0c}{a}$ and $r_2=a-\dfrac{x_0c}{a}=\dfrac{a^2-x_0c}{a}$. Also we have distances from F_1 and F_2 to tangent line $\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$:

$$F_1K=\frac{\left|\dfrac{x_0\left(-c\right)}{a^2}+\dfrac{y_0\cdot 0}{b^2}-1\right|}{\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{\left|\dfrac{x_0c}{a^2}+1\right|}{\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{\left|a+\dfrac{x_0c}{a}\right|}{a\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{a+\dfrac{x_0c}{a}}{a\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}$$

.

and

$$F_2L=\frac{\left|\dfrac{x_0c}{a^2}+\dfrac{y_0\cdot 0}{b^2}-1\right|}{\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{\left|1-\dfrac{x_0c}{a^2}\right|}{\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{\left|a-\dfrac{x_0c}{a}\right|}{a\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}=\frac{a-\dfrac{x_0c}{a}}{a^2\sqrt{\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}}}$$

because $|x_0| \le a$ and $c < a$.

Since $F_1P = r_1 = a + \dfrac{cx_0}{a}$, $F_2P = r_2 = a - \dfrac{cx_0}{a}$ and $\dfrac{x_0^2}{a^4} + \dfrac{y_0^2}{b^4} = \dfrac{1}{b^2}\left(\dfrac{b^2x_0^2}{a^4} + \dfrac{y_0^2}{b^2}\right) = \dfrac{1}{b^2}\left(\dfrac{b^2x_0^2}{a^4} + 1 - \dfrac{x_0^2}{a^2}\right) = \dfrac{1}{b^2}\left(1 + \dfrac{x_0^2}{a^4}\left(b^2 - a^2\right)\right) = \dfrac{1}{b^2}\left(1 - \dfrac{x_0^2c^2}{a^4}\right) = \dfrac{1}{a^2b^2}\left(a^2 - \dfrac{x_0^2c^2}{a^2}\right) = \dfrac{r_1r_2}{a^2b^2}$ then $\sqrt{\dfrac{x_0^2}{a^4} + \dfrac{y_0^2}{b^4}} = \dfrac{\sqrt{r_1r_2}}{ab}$ and, therefore,

$$F_1K = b\sqrt{\frac{r_1}{r_2}}, F_2L = b\sqrt{\frac{r_2}{r_1}}.$$

Hence, $\sin\alpha = \dfrac{F_1K}{r_1} = \dfrac{b}{\sqrt{r_1r_2}} = \dfrac{F_2L}{r_2} = \sin\beta$ and $F_1K \cdot F_2L = b^2$.

Another proof using that line which perpendicular to tangent line at $P \in \mathcal{E}$ is bisector of $\angle F_1PF_2$,that is using ratio $\dfrac{r_1}{r_2}$.

Vector $\mathbf{n} = \left(\dfrac{x_0}{a^2}, \dfrac{y_0}{b^2}\right)$ which perpendicular to tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at point $P(x_0, y_0)$ we call normal to ellipse at that point.

We will prove that normal is bisector of angle $\angle F_1QF_2$.

Indeed, line $\dfrac{x - x_0}{\frac{x_0}{a^2}} = \dfrac{y - y_0}{\frac{y_0}{b^2}}$ intersect OX in the point K with coordinate x which satisfy

$$\frac{x - x_0}{\frac{x_0}{a^2}} = \frac{0 - y_0}{\frac{y_0}{b^2}} \iff x = \frac{x_0\left(a^2 - b^2\right)}{a^2} = \frac{x_0c^2}{a^2}.$$

Hence,

$$F_1K = \frac{x_0c^2}{a^2} - (-c) = \frac{c\left(cx_0 + a^2\right)}{a^2},$$

$$F_2K = c - \frac{x_0c^2}{a^2} = \frac{c\left(a^2 - cx_0\right)}{a^2}.$$

Since $\dfrac{r_1}{r_2} = \dfrac{a + \frac{cx_0}{a}}{a - \frac{cx_0}{a}} = \dfrac{a^2 + cx_0}{a^2 - cx_0} = \dfrac{F_1K}{F_2K}$ then QK is bisector of angle $\angle F_1QF_2$ and, therefore, angles between tangent and radii are equal.

Remark.

Let line l be tangent to ellipse $\mathcal{E}$ at point P and let M be any point on l. Then $\min\limits_{M\in\mathcal{E}}(F_1M + F_2M) = F_1P + F_2P = 2a$.

Pic.?

0.3.2 Distance between point and ellipse.

Let $P(x_0, y_0)$ be point on the plane $\boldsymbol{P}$ laying beyond ellipse $\mathcal{E} : \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$,that is $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} > 1$

Distance between P and ellipse E by definition is $dist(P, \mathcal{E}) = \min\{PM \mid M \in E\}$,that is

$$dist(P, \mathcal{E}) = \sqrt{\min\left\{(x-x_0)^2 + (y-y_0)^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x, y \in \mathbb{R}\right\}}$$

.

We will prove that minimum of PM where $M \in \mathcal{E}$ can be attained in the point $Q(p, q) \in \mathcal{E}$ such that normal to $\mathcal{E}$ at point $Q(p, q)$ is collinear with $\overrightarrow{PQ}$.

Since equation of tangent to the ellipse at point (p, q) is $\dfrac{px}{a^2} + \dfrac{qy}{b^2} = 1$ then $\mathbf{n}\left(\dfrac{p}{a^2}, \dfrac{q}{b^2}\right)$ is normal to ellipse at point (p, q).

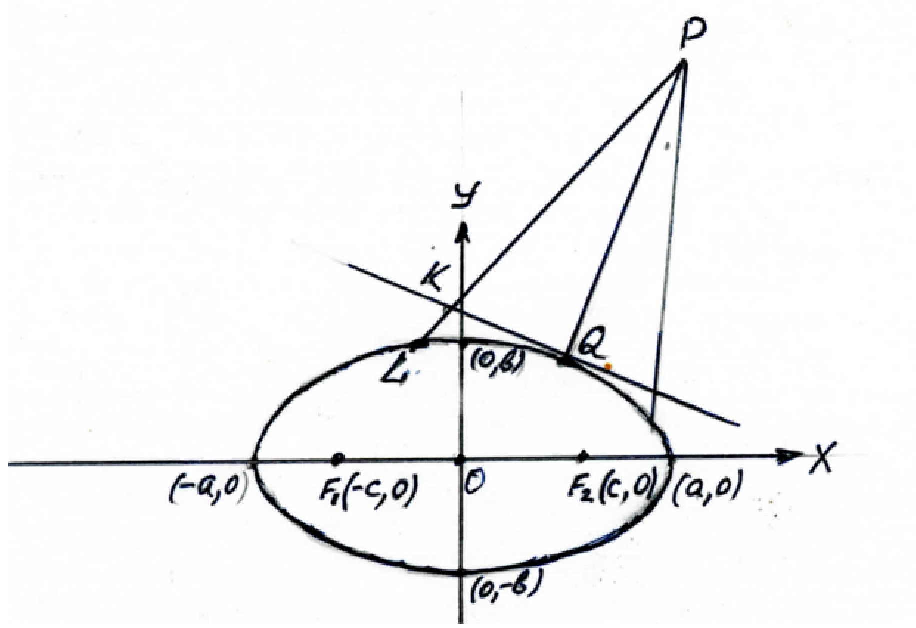
We claim $(x_0 - p, y_0 - q) = k\left(\dfrac{p}{a^2}, \dfrac{q}{b^2}\right) \iff \begin{cases} x_0 - p = \dfrac{kp}{a^2} \\ y_0 - q = \dfrac{kq}{b^2} \end{cases} \iff \begin{cases} p = \dfrac{a^2 x_0}{a^2 + k} \\ q = \dfrac{b^2 y_0}{b^2 + k} \end{cases}$.

Then $PQ^2 = (p-x_0)^2+(q-y_0)^2 = k^2\left(\dfrac{p^2}{a^4} + \dfrac{q^2}{b^4}\right) = k^2\left(\dfrac{x_0^2}{(a^2+k)^2} + \dfrac{y_0^2}{(b^2+k)^2}\right)$

where value of k can be obtained by substitution (p, q) in equation $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2} = 1$ that is from equation

$$\left(\frac{a^2 x_0}{a^2+k}\right)^2 \cdot \frac{1}{a^2} + \left(\frac{b^2 y_0}{b^2+k}\right)^2 \cdot \frac{1}{b^2} = 1 \iff \frac{a^2 x_0^2}{(a^2+k)^2} + \frac{b^2 y_0^2}{(b^2+k)^2} = 1.$$

Thus, $PQ = |k|\sqrt{\dfrac{p^2}{a^4} + \dfrac{q^2}{b^4}} = |k|\sqrt{\dfrac{x_0^2}{(a^2+k)^2} + \dfrac{y_0^2}{(b^2+k)^2}}$



Let $L \neq Q$ be any point on ellipse then $PL > PK \geq PQ$ and, therefore, point Q realize minimum distance.

This is geometric visual proof can be confirmed with the following analytic proof based on using partial derivatives and Lagrange multipliers.

Namely, let $F(x,y,t) := (x-x_0)^2 + (y-y_0)^2 - t\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right)$. Then we have system of equation

$F'_x(x,y,t) = 2\left(x - x_0 - \dfrac{tx}{a^2}\right) = 0, F'_y(x,y,t) = 2\left(y - y_0 - \dfrac{ty}{b^2}\right) = 0$ and $F'_t(x,y,t) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0.$

Which give us the same result as above.

0.3.3 Locus of equidistant points.

1.Parametric representation.

Let point $P(x,y)$ belong to locus of equidistant points for ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, that is $dist(P,\mathcal{E}) = d$, where d is given positive constant and let $Q(p,q)$ point on ellipse $\mathcal{E}$ such that $PQ = d$.Then $(x-p, y-q) \parallel \left(\dfrac{p}{a^2}, \dfrac{q}{b^2}\right)$ and equation of line PQ is $\dfrac{x-p}{\dfrac{p}{a^2}} = \dfrac{y-q}{\dfrac{q}{b^2}}$.

Let $t=\dfrac{x-p}{\dfrac{p}{a^2}}=\dfrac{y-q}{\dfrac{q}{b^2}}$ then $p=\dfrac{a^2x}{a^2+t},q=\dfrac{b^2y}{b^2+t}$.

Therefore, $d^2=(x-p)^2+(y-q)^2=\left(x-\dfrac{a^2x}{a^2+t}\right)^2+\left(y-\dfrac{b^2y}{b^2+t}\right)^2 \iff$

$\dfrac{d^2}{t^2}=\dfrac{x^2}{(a^2+t)^2}+\dfrac{y^2}{(b^2+t)^2}$ and

$\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1 \iff \dfrac{a^2x^2}{(a^2+t)^2}+\dfrac{b^2y^2}{(b^2+t)^2}=1.$

Thus, $\begin{cases}\dfrac{x^2}{(a^2+t)^2}+\dfrac{y^2}{(b^2+t)^2}=\dfrac{d^2}{t^2}\\ \dfrac{a^2x^2}{(a^2+t)^2}+\dfrac{b^2y^2}{(b^2+t)^2}=1\end{cases} \iff \begin{cases}\dfrac{b^2d^2}{t^2}-1=\dfrac{(b^2-a^2)x^2}{(a^2+t)^2}\\ \dfrac{a^2d^2}{t^2}-1=\dfrac{(a^2-b^2)y^2}{(b^2+t)^2}\end{cases} \iff$

$$\begin{cases}\dfrac{b^2d^2}{t^2}-1=-\dfrac{c^2x^2}{(a^2+t)^2}\\ \dfrac{a^2d^2}{t^2}-1=\dfrac{c^2y^2}{(b^2+t)^2}\end{cases}.$$

If $c\neq 0$ then (!) $\iff \begin{cases}\dfrac{b^2d^2}{t^2}-1=-\dfrac{c^2x^2}{(a^2+t)^2}\\ \dfrac{a^2d^2}{t^2}-1=\dfrac{c^2y^2}{(b^2+t)^2}\\ \dfrac{b^2d^2}{t^2}\leq 1,\dfrac{a^2d^2}{t^2}\geq 1\end{cases} \iff \begin{cases}x^2=\dfrac{(t^2-b^2d^2)(a^2+t)^2}{t^2c^2}\\ y^2=\dfrac{(a^2d^2-t^2)(b^2+t)^2}{t^2c^2}\\ bd\leq |t|\leq ad\end{cases}.$

If $c=0$ then $a=b$, $\dfrac{a^2d^2}{t^2}-1=0$ and $\begin{cases}\dfrac{x^2}{(a^2+t)^2}+\dfrac{y^2}{(b^2+t)^2}=\dfrac{d^2}{t^2}\\ \dfrac{a^2x^2}{(a^2+t)^2}+\dfrac{b^2y^2}{(b^2+t)^2}=1\end{cases} \iff$

$\dfrac{a^2x^2}{(a^2+t)^2}+\dfrac{a^2y^2}{(a^2+t)^2}=1 \iff x^2+y^2=\dfrac{(a^2+t)^2}{a^2} \iff$

$x^2+y^2=\dfrac{(a^2\pm ad)^2}{a^2} \iff x^2+y^2=(a\pm d)^2.$

0.3.4 Distance between line and ellipse.

Problem.

Find distance between line and ellipse where ellipse $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and line $l: px+qy=r$ have no common points.

First find out for what values a,b,p,q,r ellipse $\mathcal{E}$ and line l have no common points, that is when system

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ px + qy = r \end{cases}$$

have no solution.

Since $(px+qy)^2 + \left(\dfrac{qbx}{a} - \dfrac{pay}{b}\right)^2 = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)(p^2a^2 + q^2b^2) \iff r^2 + \left(\dfrac{qbx}{a} - \dfrac{pay}{b}\right)^2 = p^2a^2 + q^2b^2 \iff \left(\dfrac{qbx}{a} - \dfrac{pay}{b}\right)^2 = p^2a^2 + q^2b^2 - r^2$ then $p^2a^2 + q^2b^2 < r^2$ because otherwise if $p^2a^2 + q^2b^2 \geq r^2$ then $\dfrac{qbx}{a} - \dfrac{pay}{b} = \pm\sqrt{p^2a^2 + q^2b^2 - r^2}$ and we obtain linear system

$$\begin{cases} px + qy = r \\ \dfrac{qbx}{a} - \dfrac{pay}{b} = \pm\sqrt{p^2a^2 + q^2b^2 - r^2} \end{cases}$$

which always has solution (determinant equal $\dfrac{p^2a}{b} + \dfrac{q^2b}{a} = \dfrac{p^2a^2 + q^2b^2}{ab} > 0.$).

Thus, the ellipse $\mathcal{E}$ and the line l have no common points iff $p^2a^2 + q^2b^2 < r^2$.

Let $M(x,y)$ be any point on ellipse. Then $dist(M,L) = \dfrac{|px+qy-r|}{\sqrt{p^2+q^2}}$ and $dist(\mathcal{E}, l) = \min\limits_{x,y} \dfrac{|px+qy-r|}{\sqrt{p^2+q^2}}$, where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Since $px + qy = pa \cdot \dfrac{x}{a} + qb \cdot \dfrac{y}{b}$ then by Cauchy Inequality we have

(C) $\qquad px + qy \leq \sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}} \cdot \sqrt{p^2a^2 + q^2b^2} = \sqrt{p^2a^2 + q^2b^2}$.

Since $r - (px+qy) \geq r - \sqrt{p^2a^2 + q^2b^2} > 0$ then $|r - (px+qy)| = r - (px+qy) \geq r - \sqrt{p^2a^2 + q^2b^2}$, where lower bound $r - \sqrt{p^2a^2 + q^2b^2}$ for $|r - (px+qy)|$ can be attained, because $\left(\dfrac{x}{a}, \dfrac{y}{b}\right) = k(pa, qb) \iff (x,y) = (kpa^2, kqb^2), k > 0$ (condition of equality in inequality **(C)**) together with claim $M(x,y) \in \mathcal{E}$ give us

$$\frac{(kpa^2)^2}{a^2} + \frac{(kqb^2)^2}{b^2} = 1 \iff k = \frac{1}{\sqrt{p^2a^2 + q^2b^2}}.$$

Let $(x_*, y_*) := \left(\dfrac{pa^2}{\sqrt{p^2a^2 + q^2b^2}}, \dfrac{qb^2}{\sqrt{p^2a^2 + q^2b^2}}\right)$.Then point $M(x_*, y_*) \in \mathcal{E}$ and $|r - (px_* + qy_*)| = \left|r - \sqrt{p^2a^2 + q^2b^2}\right| = r - \sqrt{p^2a^2 + q^2b^2}$ and, therefore,

$$dist(\mathcal{E}, l) = \min \frac{|px+qy-r|}{\sqrt{p^2+q^2}} = \frac{r - \sqrt{p^2a^2 + q^2b^2}}{\sqrt{p^2+q^2}}$$

.

0.3.5 Angle of observation of ellipse from given exterior point.

Let $P(x_0, y_0)$ be a point beyond the ellipse $\mathcal{E}$: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, that is $\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}>1$.

Find angle of observation of ellipse from the point P, that is an angle between tangents from the point P to ellipse $\mathcal{E}$.

Let L be the line that tangent to the ellipse $\mathcal{E}$ at point Q.The situation determined by two conditions:

$Q(p,q)\in\mathcal{E}$ and $P(x_0,y_0)\in l$, that is,

(1) $\begin{cases}\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1\\ \dfrac{px_0}{a^2}+\dfrac{qy_0}{b^2}=1\end{cases}$.

(or, $(x_0-p, y_0-q)=t\left(\dfrac{p}{a^2},\dfrac{q}{b^2}\right)$ and $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1$).

Denoting $u:=\dfrac{p}{a}, v:=\dfrac{q}{b}$ and $x_1:=\dfrac{x_0}{a}, y_1:=\dfrac{y_0}{b}$ we can rewrite **(1)** as

(2) $\begin{cases}u^2+v^2=1\\ ux_1+vy_1=1\end{cases}$ and solve it in u, v.

Since $x_1^2+y_1^2>1$ and $(ux_1+vy_1)^2+(uy_1-vx_1)^2=\left(u^2+v^2\right)\left(x_1^2+y_1^2\right)$ then $1+(uy_1-vx_1)^2=x_1^2+y_1^2 \iff |uy_1-vx_1|=d$, where $d:=\sqrt{x_1^2+y_1^2-1}$ and (2) $\iff$

(3) $\begin{cases}|uy_1-vx_1|=d\\ ux_1+vy_1=1\end{cases}$.

Solving the system **(3)** we obtain two solutions (because $d>0$):

$(u_1,v_1)=\left(\dfrac{x_1+dy_1}{1+d^2},\dfrac{y_1-dx_1}{1+d^2}\right)$ and $(u_2,v_2)=\left(\dfrac{x_1-dy_1}{1+d^2},\dfrac{y_1+dx_1}{1+d^2}\right)$.

Coming back to original notation we get

$(p_1,q_1)=(au_1,bv_1)=\left(\dfrac{a\left(\dfrac{x_0}{a}+d\dfrac{y_0}{b}\right)}{1+d^2},\dfrac{b\left(\dfrac{y_0}{b}-d\dfrac{x_0}{a}\right)}{1+d^2}\right)=$

$\left(\dfrac{bx_0+day_0}{b(1+d^2)},\dfrac{ay_0-dbx_0}{a(1+d^2)}\right)$, $(p_2,q_2)=\left(\dfrac{bx_0-day_0}{b(1+d^2)},\dfrac{ay_0+dbx_0}{a(1+d^2)}\right)$ and

$(x_0-p_1,y_0-q_1)=\left(x_0-\dfrac{bx_0+day_0}{b(1+d^2)},y_0-\dfrac{ay_0-dbx_0}{a(1+d^2)}\right)=$

$\left(\dfrac{d(bdx_0-ay_0)}{b(d^2+1)},\dfrac{d(ady_0+bx_0)}{a(d^2+1)}\right)=\dfrac{d\mathbf{t}_1}{ab(d^2+1)}$, where $\mathbf{t}_1:=(a(bdx_0-ay_0), b(ady_0+bx_0))$,

and $(x_0-p_2,y_0-q_2)=\left(\dfrac{d(bdx_0+ay_0)}{b(d^2+1)},\dfrac{d(ady_0-bx_0)}{a(d^2+1)}\right)=\dfrac{d\mathbf{t}_2}{ab(d^2+1)}$,

where $\mathbf{t}_2:=(a(bdx_0+ay_0), b(ady_0-bx_0))$.

Let $\alpha:=\widehat{\mathbf{t}_1,\mathbf{t}_2}$. Since $\cos\alpha=\dfrac{\mathbf{t}_1\cdot\mathbf{t}_2}{\|\mathbf{t}_1\|\,\|\mathbf{t}_2\|}$ and $\sin\alpha=\dfrac{\mathbf{t}_1\wedge\mathbf{t}_2}{\|\mathbf{t}_1\|\,\|\mathbf{t}_2\|}$ then $\cot\alpha=\dfrac{\mathbf{t}_1\cdot\mathbf{t}_2}{\mathbf{t}_1\wedge\mathbf{t}_2}$.

We have $\mathbf{t}_1\cdot\mathbf{t}_2=a^2\left(b^2d^2x_0^2-a^2y_0^2\right)+b^2\left(a^2d^2y_0^2-b^2x_0^2\right)=$

$-\left(a^4y_0^2+b^4x_0^2-\left(a^2b^2d^2\left(x_0^2+y_0^2\right)\right)\right)=$

$a^4y_0^2+b^4x_0^2-\left(a^2b^2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-1\right)\left(x_0^2+y_0^2\right)\right)=\left(a^2+b^2-x_0^2-y_0^2\right)\left(a^2y_0^2+b^2x_0^2\right).$

$\mathbf{t}_1\wedge\mathbf{t}_2=\det\begin{pmatrix} a\left(bdx_0-ay_0\right) & b\left(ady_0+bx_0\right)\\ a\left(bdx_0+ay_0\right) & b\left(ady_0-bx_0\right)\end{pmatrix}=$

$ab\left(bdx_0-ay_0\right)\left(ady_0-bx_0\right)-ab\left(ady_0+bx_0\right)\left(bdx_0+ay_0\right)=-2abd\left(a^2y_0^2+b^2x_0^2\right)$

and, therefore,

(4) $\cot\alpha=\dfrac{a^2+b^2-x_0^2-y_0^2}{2abd}.$

($\cot\alpha$ is best way to determine α because $range\left(\cot^{-1}\alpha\right)=(0,\pi)$ .Calculations α, using

formula $\cos\alpha=\dfrac{\mathbf{t}_1\cdot\mathbf{t}_2}{\|\mathbf{t}_1\|\,\|\mathbf{t}_2\|}$ leads to ponderous work).

In the case when α is angle of observation of ellipse $\mathcal{E}$ from exterior point P we saying that P is point of of $\alpha-$observation .

Locus of all $\alpha-$observation points for given ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

For given α by formula **(4)** we obtain $a^2+b^2-x_0^2-y_0^2=2abd\cot\alpha$.

Since $d=\sqrt{\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}-1}$ we obtain the following equation for locus:

(5) $a^2+b^2=x^2+y^2+2ab\cot\alpha\sqrt{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1}.$

In the case $\alpha=\dfrac{\pi}{2}$ equation becomes equation of the circle

$x^2+y^2=a^2+b^2.$

Parametric representation of ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

1. **Trigonometric parametrization.**

Let (x,y) is solution of equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.Then point $\left(\dfrac{x}{a},\dfrac{y}{b}\right)$ lie on the unite circle and, therefore, $\left(\dfrac{x}{a},\dfrac{y}{b}\right)=(\cos t,\sin t)\iff(x,y)=(a\cos t,b\sin t)\,,t\in[0,2\pi)$.

2. **Algebraic (rational) parametrization.**

Since any (u,v) such that $u^2+v^2=1$ can be represented in the form

$\left(\dfrac{1-t^2}{1+t^2},\dfrac{2t}{1+t^2}\right),t\in\mathbb{R}$

then $\left(\dfrac{x}{a},\dfrac{y}{b}\right)=\left(\dfrac{1-t^2}{1+t^2},\dfrac{2t}{1+t^2}\right)\iff(x,y)=\left(\dfrac{a\left(1-t^2\right)}{1+t^2},\dfrac{2bt}{1+t^2}\right).$

Polar equation of Ellipse.

As we know (see "Reflection property of Ellipse") for any point $P(x,y)$ on ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ we have $r_1=a+\dfrac{xc}{a}=a+ex,r_2=a-x_0e$,where $0<e=\dfrac{c}{a}<1$ is eccentricity of ellipse $\mathcal{E}$. Let polar origin coincide with $F_2(c,0\,)$.

Let $r := r_2 = a - xe$ be polar radius and θ be angle between OX and F_2P counter counterclockwise. Then $x = c + r\cos\theta$ and since $xe = a - r$ we obtain $ec + er\cos\theta = a - r \iff r = \dfrac{a - ec}{1 + e\cos\theta}$. Since $a - ec = a\left(1 - e\dfrac{c}{a}\right) = a\left(1 - e^2\right)$ and $a - ec = a - \dfrac{c}{a}\cdot c = \dfrac{b^2}{a}$ then we also can use $r = \dfrac{a\left(1 - e^2\right)}{1 + e\cos\theta}$ or $r = \dfrac{b^2}{a\left(1 + e\cos\theta\right)}$ or $r = \dfrac{b^2}{a + c\cos\theta}, \theta \in [0, 2\pi)$.

Equation of ellipse in general position on the coordinate plane.

1. First we consider ellipse $\mathcal{E}$ given by equation $\dfrac{(x-p)^2}{a^2} + \dfrac{(y-q)^2}{b^2} = 1$,where p, q be given real numbers.

Let $u := x - p, v := y - q$ and we consider new coordinate axes represented in the coordinate system XOY by lines $x = p$ (axe O_1V), $y = q$ (axe O_1U) where $O_1\left(p, q\right)$ is new origin. Equation of $\mathcal{E}$ in coordinate system UO_1V becomes $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$ with foci coordinates $(-c, 0)$ and $(c, 0)$.Hence, foci coordinates in coordinate system OXY are $(-c + p, q)$ and $(c + p, q)$.

2. Let $\mathcal{K}$ be set of points on the coordinate plane XOY defined as follows

$$\mathcal{K} := \left\{(x, y) \mid x, y \in \mathbb{R} \ \&\ ax^2 + 2bxy + cy^2 = d\right\}$$

where $d > 0$ and $ax^2 + 2bxy + cy^2$ be positively defined Quadratic Form

(homogeneous polynomial $F\left(x, y\right)$ of the second degree is positively defined iff $F\left(x, y\right) \geq 0$ for any real x, y and $F\left(x, y\right) = 0 \iff x = y = 0$).

Note that $\mathcal{K}$ is central symmetric because if $(x, y) \in \mathcal{K}$ then $(-x, -y) \in \mathcal{K}$ as well.

We will prove that $F\left(x, y\right) = ax^2 + 2bxy + cy^2$ is positively defined iff $a > 0$ and $ac - b^2 > 0$.

By considering case $b = 0$ as trivial (because then obvious that $F\left(x, y\right) = ax^2 + cy^2$ is positively defined iff $a, c > 0$ and, therefore, equation $ax^2 + cy^2 = d \iff \dfrac{x^2}{\left(d/a\right)^2} + \dfrac{y^2}{\left(d/b\right)^2} = 1$ represent ellipse in the standard position) for further we assume that $b \neq 0$.

Since $a^2 + b^2 \neq 0$ (because otherwise $F\left(x, y\right) = 2bxy$ isn't positively defined

WLOG we can assume that $a \neq 0$. Since $F\left(x, y\right) = \dfrac{1}{a}\left(\left(ax + by\right)^2 + \left(ac - b^2\right)y^2\right) \geq 0$ for any $x, y \in \mathbb{R}$ then $a > 0$ and $ac - b^2 > 0$.Indeed, since $F\left(-\dfrac{by}{a}, y\right) = \dfrac{\left(ac - b^2\right)y^2}{a} \geq 0 \iff \dfrac{ac - b^2}{a} \geq 0$ for any $y \neq 0$ and $F\left(x, 0\right) = ax^2 > 0 \iff a > 0$ for any $x \neq 0$ then $a > 0$ and $ac - b^2 > 0$ (hence, $c > 0$ as well).

And vice versa if $a > 0$ and $ac - b^2 > 0$ then $F\left(x, y\right)$ is positively defined.

Indeed, $F\left(x, y\right) = \dfrac{1}{a}\left(\left(ax + by\right)^2 + \left(ac - b^2\right)y^2\right) \geq 0$ for any $x, y \in \mathbb{R}$ and

$$F(x,y)=0 \iff (ax+by)^2+(ac-b^2)\,y^2=0 \iff \begin{cases} ax+by=0 \\ (ac-b^2)\,y^2=0 \end{cases} \iff x=y=0.$$

We will prove that equation $ax^2+2bxy+cy^2=d,$ where $d,a,ac-b^2>0$ and is equation of ellipse, which reduced to the standard form in new coordinate system UOV obtained by rotation coordinate system XOY on some angle φ.

In coordinate system XOY two orthonormal vectors $\mathbf{i}(1,0)$ and $\mathbf{j}\,(0,1)$ form basis, that is any vector $\mathbf{z}$ have unique representation as linear combination of vectors $\mathbf{i}$ and $\mathbf{j}$ namely $\mathbf{z}=x\mathbf{i}+y\mathbf{j}$ and pair of coefficients (x,y) is coordinate of $\mathbf{z}$ in coordinate system XOY.Let $\mathbf{i}_1$ and $\mathbf{j}_1$ be two orthonormal vectors formed by rotation of vectors $\mathbf{i}$ and $\mathbf{j}$ on angle φ counterclockwise. Then $\mathbf{i}_1=\cos\varphi{\cdot}\mathbf{i}+\sin\varphi{\cdot}\mathbf{j}$ and $\mathbf{j}_1=-\sin\varphi{\cdot}\mathbf{i}+\cos\varphi{\cdot}\mathbf{j}$.

Hence $\mathbf{i}=\cos\varphi\cdot\mathbf{i}_1-\sin\varphi\cdot\mathbf{j}_1, \mathbf{j}=\sin\varphi\cdot\mathbf{i}_1+\cos\varphi\cdot\mathbf{j}_1$.

Let UOV be new Cartesian system of coordinates which correspondent to basis $\mathbf{i}_1,\mathbf{j}_1$ (shortly $UOV=R_\varphi(XOZ)$ where R_φ is operator of rotation of plane $\mathcal{P}$ with respect to origin O on angle φ) and and let arbitrary vector $\mathbf{z}$ has coordinates (x,y) in coordinate system XOY and coordinates (u,v) in coordinate system UOV.

Then $\mathbf{z}=u\mathbf{i}_1+v\mathbf{j}_1 \;=u\,(\cos\varphi{\cdot}\mathbf{i}+\sin\varphi{\cdot}\mathbf{j})+v\,(-\sin\varphi\cdot\mathbf{i}+\cos\varphi\cdot\mathbf{j}) \;=(u\cos\varphi-v\sin\varphi)\,\mathbf{i}+(u\sin\varphi+v\cos\varphi)\,\mathbf{j}_1$ and, therefore,

$$\begin{cases} x=u\cos\varphi-v\sin\varphi \\ y=u\sin\varphi+v\cos\varphi \end{cases}.$$

Solving this system with respect to u and v we obtain

$$\begin{cases} u=x\cos\varphi+y\sin\varphi \\ v=-x\sin\varphi+y\cos\varphi \end{cases}$$

,

By substitution $x=u\cos\varphi-v\sin\varphi,\; y=u\sin\varphi+v\cos\varphi$ in equation $ax^2+2bxy+cy^2=d$ we obtain

$$a\,(u\cos\varphi-v\sin\varphi)^2+2b\,(u\cos\varphi-v\sin\varphi)\,(u\sin\varphi+v\cos\varphi)+c\,(u\sin\varphi+v\cos\varphi)^2=$$
$$u^2F(\cos\varphi,\sin\varphi)+v^2F(-\sin\varphi,\cos\varphi)+uv\left(2\,(c-a)\sin\varphi\cos\varphi+2b\left(\cos^2\varphi-\sin^2\varphi\right)\right).$$

We claim $2\,(c-a)\sin\varphi\cos\varphi+2b\left(\cos^2\varphi-\sin^2\varphi\right)=0 \iff 2b\cos2\varphi=(a-c)\sin2\varphi \iff$

$$\cot2\varphi=\frac{a-c}{2b} \iff \varphi=\varphi_*:=\frac{1}{2}\operatorname{arccot}\frac{a-c}{2b}.$$

Thus, $ax^2+2bxy+cy^2=d \iff pu^2+qv^2=d$,where $p:=F(\cos\varphi_*,\sin\varphi_*)>0$ and $q:=F(-\sin\varphi_*,\cos\varphi_*)>0$ since $F(\cos\varphi,\sin\varphi)>0$ for any φ.

Therefore, equation $F(x,y)=d, d>0$ is equation of ellipse with center of symmetry in origin $O(0,0)$ and axis of symmetry $u=0 \iff x\cos\varphi_*+y\sin\varphi_* \iff y=-x\cot\varphi_*$ and $v=0 \iff -x\sin\varphi_*+y\cos\varphi_*=0 \iff y=x\tan\varphi_*$.

Consider now polynomial of the second degree $H(x,y)=F(x,y)+2ex+2fy$,where $F(x,y):=ax^2+2bxy+cy^2$.

We will try to find numbers p and q such that $H(u+p,v+q)$ can be represented as a sum of quadratic form with respect to u and v and some constant. For that in expression

$H(u+p,v+q) = a(u+p)^2 + 2b(u+p)(v+q) + c(v+q)^2 + 2e(u+p) + 2f(v+q) =$

$au^2 + 2buv + cv^2 + 2u(e+ap+bq) + 2v(f+bp+cq) + ap^2 + 2bpq + cq^2 + 2ep + 2fq$

we claim $e+ap+bq = 0$ and $f+bp+cq = 0$.This system of equations unique determine p,q iff $\det\begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0$.

Since, in the case when quadratic form $F(x,y)$ is positively defined this claim is fulfilled then for obtained by such way p and q we have

$H(u+p,v+q) = F(u,v) + H(p,q) \iff$

$H(x,y) = F(x-p,y-p) + F(p,q) + 2ep + 2fq$

and equation $H(x,y) = d \iff F(x,y) + 2ex + 2fy = d$ represent ellipse on coordinate plane XOY in general position if $d > F(p,q) + 2ep + 2fq$.

Remark.

In the case of equation $ax^2 + 2bxy + cy^2 = d$,where $d.a, ac - b^2 > 0$ we can find axes of symmetry without references to vectors and rotation of coordinate system.

Namely we can use the following property of ellipse:

Both axes of symmetry of the ellipse by their intersection with ellipse points give us biggest and smallest diameters of the ellipse. (Diameter of the ellipse we call any

segment with the end on the ellipse and which contain center of ellipse).

Since $\begin{cases} y = kx \\ ax^2 + 2bxy + cy^2 = d \end{cases} \iff \begin{cases} x^2 = \dfrac{d}{a + 2bk + ck^2} \\ y = kx \end{cases}$

then square of length of diameter with slope k is $4\left(x^2 + k^2x^2\right) = 4x^2\left(1+k^2\right) = \dfrac{4d\left(1+k^2\right)}{a + 2bk + ck^2}$.

Since $k = \tan\varphi$ then $\dfrac{4d\left(1+k^2\right)}{a + 2bk + ck^2} = \dfrac{4d}{a\cos^2\varphi + 2b\cos\varphi\sin\varphi + c\sin^2\varphi}$ then remains to find φ for which $a\cos^2\varphi + 2b\cos\varphi\sin\varphi + c\sin^2\varphi$ attain maximal and minimal values.

We have $2\left(a\cos^2\varphi + 2b\cos\varphi\sin\varphi + c\sin^2\varphi\right) = a(1+\cos 2\varphi) + c(1-\cos 2\varphi) + 2b\sin 2\varphi =$

$a + c + (a-c)\cos 2\varphi + 2b\sin 2\varphi.$

Since by Cauchy Inequality $|(a-c)\cos 2\varphi + 2b\sin 2\varphi| \leq \sqrt{(a-c)^2 + 4b^2}$ and equality occurs iff $(a-c)\sin 2\varphi = 2b\cdot\cos 2\varphi \iff \cot 2\varphi = \dfrac{a-c}{2b}$.Solving latter equation for $\varphi \in (0,\pi)$ we obtain two values for 2φ,namely $2\varphi \in \operatorname{arccot}\dfrac{a-c}{2b} \iff \varphi = \dfrac{1}{2}\operatorname{arccot}\dfrac{a-c}{2b}$ and $2\varphi \in \pi + \operatorname{arccot}\dfrac{a-c}{2b} \iff \varphi = \dfrac{\pi}{2} + \dfrac{1}{2}\operatorname{arccot}\dfrac{a-c}{2b}$.

0.4 Problems.

Problem 1. (Chords in ellipse).

Prove that midpoints of parallel chords with ends that belong to ellipse formed the line $y = mx$ and find its slope m.

Problem 2.

Let point $P(x_0, y_0)$ be exterior with respect to ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.Find equation of line which passed through P and tangent to ellipse $\mathcal{E}$.

Problem 3.

Find necessary and sufficient condition for numbers p, q, r that the line $l : px + qy = r$ will be tangent to ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Problem 4.

Find equation of tangent line to ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which:

a) Is parallel to line $l : px + qy = r$;
b) Is perpendicular to $l : px + qy = r$;
c) Formed given angle α,counted counterclockwise, with line $l : px + qy = r$.

Problem 5.

Let F_1, F_2 be the foci of an ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Prove that product of distances from F_1, F_2 to any line that tangent to $\mathcal{E}$ is equal to b^2.

Problem 6.

Among all rectangle inscribed in the ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with sides that parallel to its axes find the rectangle with maximum area.

Problem 7.

Find the maximal area of a triangle ABC inscribed in ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ so, that $BC \parallel \overleftrightarrow{OX}$.

0.4.1 Solutions.

Solution to Problem 1.

Let AB is one of the family of a parallel chords with slope k on the ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and let $(x_1, y_1), (x_2, y_2)$ be coordinates of A, B, respectively.

Let (x, y) be coordinates of midpoint of chord AB, that is $x = \dfrac{x_1 + x_2}{2}, y = \dfrac{y_1 + y_2}{2}$.

Since $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$, $\dfrac{x_2^2}{a^2} + \dfrac{y_2^2}{b^2} = 1$ and $\dfrac{y_2 - y_1}{x_2 - x_1} = k$ then $\dfrac{x_1^2 - x_2^2}{a^2} + \dfrac{y_1^2 - y_2^2}{b^2} = 0 \iff$

$$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} + \frac{(y_1 - y_2)(y_1 + y_2)}{b^2} = 0 \iff$$

$$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} + \frac{k(x_1 - x_2)(y_1 + y_2)}{b^2} = 0 \iff \frac{x_1 + x_2}{a^2} + \frac{k(y_1 + y_2)}{b^2} = 0 \iff$$

$$\frac{x}{a^2} + \frac{ky}{b^2} = 0 \iff y = -\frac{b^2}{ka^2}x.$$ Hence, $m = -\dfrac{b^2}{ka^2}$.

Solution to Problem 2.

Point $P(x_0, y_0)$ is exterior with respect to ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ iff $\delta := \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}-1>0$.

Let $M(x_1, y_1)$ be point on ellipse $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, that is $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}=1$.

Then $\overleftrightarrow{PM}:\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$ is tangent to $\mathcal{E}$ and $\dfrac{x_0x_1}{a^2}+\dfrac{y_0y_1}{b^2}=1$.

Thus, we obtain the following system of equation with respect to x_1, y_1 :

$$(!)\qquad \begin{cases} \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}=1 \\ \dfrac{x_0x_1}{a^2}+\dfrac{y_0y_1}{b^2}=1 \end{cases}.$$

Applying Lagrange Identity to pairs $\left(\dfrac{x_0}{a},\dfrac{y_0}{b}\right),\left(\dfrac{x_1}{a},\dfrac{y_1}{b}\right)$ we obtain

$$\left(\frac{x_0x_1}{a^2}+\frac{y_0y_1}{b^2}\right)^2+\left(\frac{x_0y_1}{ab}-\frac{y_0x_1}{ba}\right)^2=\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\iff$$

$$1+\frac{1}{a^2b^2}(x_0y_1-y_0x_1)^2=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\iff y_0x_1-y_1x_0=\pm ab\delta$$

and, therefore, $(!)\iff \begin{cases} \dfrac{x_0x_1}{a^2}+\dfrac{y_0y_1}{b^2}=1 \\ y_0x_1-y_1x_0=\pm ab\delta \end{cases}\iff \begin{cases} x_1=\dfrac{bx_0\pm a\delta y_0}{b} \\ y_1=\dfrac{ay_0\mp b\delta x_0}{a} \end{cases}.$

Solution to Problem 3.

Line $l: px+qy=r$ is tangent line to ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ iff the system

$(!!)\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \\ px+qy=r \end{cases}$ have unique solution (x,y).

Applying Lagrange Identity to pairs $(pa,qb),\left(\dfrac{x}{a},\dfrac{y}{b}\right)$ we obtain

$$\left(pa\cdot\frac{x}{a}+qb\cdot\frac{y}{b}\right)^2+\left(pa\cdot\frac{y}{b}-qb\cdot\frac{x}{a}\right)^2=\left(p^2a^2+q^2b^2\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\iff$$

$$r^2+\left(\frac{bqx}{a}-\frac{apy}{b}\right)^2=p^2a^2+q^2b^2\iff\left(\frac{bqx}{a}-\frac{apy}{b}\right)^2=p^2a^2+q^2b^2-r^2.$$

Hence, $p^2a^2+q^2b^2\geq r^2$. Since $\dfrac{bqx}{a}-\dfrac{apy}{b}=\pm\sqrt{p^2a^2+q^2b^2-r^2}$if

$p^2a^2+q^2b^2\geq r^2$ then in that case $(!!)\iff \begin{cases} px+qy=r \\ \dfrac{bqx}{a}-\dfrac{apy}{b}=\pm\sqrt{p^2a^2+q^2b^2-r^2} \end{cases}$

where latter system is linear with respect to x, y and solvable(because its determinant isn't zero).

Thus, necessary and sufficient condition for numbers p, q, r is inequality $p^2a^2+q^2b^2\geq r^2$.

Solution to Problem 4.

a) Let $P(x_0, y_0)\in\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ such that line $\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$ is parallel to line l given by equation $px+qy=r$.Then $\left(\dfrac{x_0}{a^2},\dfrac{y_0}{b^2}\right)=k(p,q) \iff (x_0,y_0)=\left(kpa^2,kqb^2\right)$,where $k\in\mathbb{R}\setminus\{0\}$ and, therefore, $P(x_0,y_0)\in\mathcal{E} \iff \dfrac{\left(kpa^2\right)^2}{a^2}+\dfrac{\left(kqb^2\right)^2}{b^2}=1 \iff k=\pm\dfrac{1}{\sqrt{p^2a^2+q^2b^2}}$.

Thus, we obtain two tangent lines $xp+qy=\dfrac{1}{k}$ where, $k=\pm\dfrac{1}{\sqrt{p^2a^2+q^2b^2}}$.

b) Since tangent line $\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$ is perpendicular to $l:px+qy=r$ then $\left(\dfrac{x_0}{a^2},\dfrac{y_0}{b^2}\right)=k(-q,p), k\in\mathbb{R}\setminus\{0\}$ and, therefore, $P(x_0,y_0)\in\mathcal{E} \iff \dfrac{\left(k(-q)a^2\right)^2}{a^2}+\dfrac{\left(kpb^2\right)^2}{b^2}=1 \iff k=\pm\dfrac{1}{\sqrt{q^2a^2+p^2b^2}}$.

Thus, we obtain two tangent lines $-qx+py=\dfrac{1}{k}$ where, $k=\pm\dfrac{1}{\sqrt{p^2a^2+q^2b^2}}$.

c) Since normal to ellipse at point $P(x_0,y_0)$ formed angle α with normal to line $l:px+qy=r$ then $\left(\dfrac{x_0}{a^2},\dfrac{y_0}{b^2}\right)=k(p_1,q_1)$, where

$$\begin{pmatrix}p_1\\q_1\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}.$$

And further as in **a)**.

Solution to Problem 5.

See the first proof of **Reflection property of tangent to ellipse.**

Problem 6.

Let $P(x,y)$,where $x,y>0$ be vertex of the rectangle inscribed in ellipse $\mathcal{E}$:$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Then area of the rectangle is $A(x,y):=4xy$ and $1=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\geq\dfrac{2xy}{ab}=\dfrac{A(x,y)}{2ab} \iff$

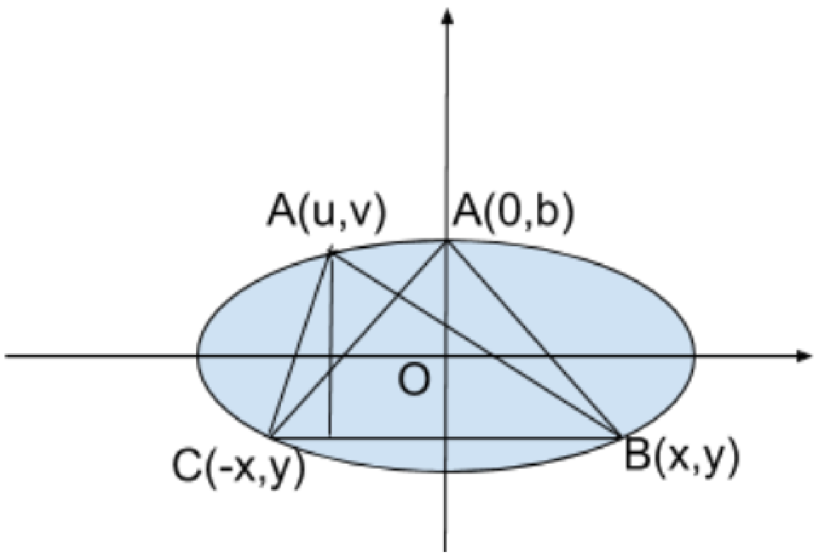
$A(x,y)\leq 2ab$.Since $A\left(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}}\right)=2ab$ then $\max A(x,y)=2ab$.

Problem 7.

Let (u,v) be coordinates of vertex A and (x,y) be coordinates of vertex B such that $x>0$.Then $(-x,y)$ be coordinates of vertex C and since $BC=2x$,altitude $h_a=v-y$ then area of $\triangle ABC$ is $[ABC]=(v-y)x$.

First note that for any such triangles with fixed vertex B the biggest area have triangles with A placed on OY that is if $(u,v)=(0,b)$ (taking into account the symmetry) and $\triangle ABC$ is isosceles triangle. Thus, we will find $\max(b-y)x$.

Using trigonometric parametrization of the ellipse in the form $(x,y)=(a\sin t,b\cos t)$, where $t\in(0,\pi)$ because $x>0$ and using AM-GM inequality we obtain



$$[ABC] = (b - b\cos t)\, a\sin t = 4ab\sin^3\frac{t}{2}\cos\frac{t}{2} = 4ab\sqrt{\frac{1}{3}\left(\sin^2\frac{t}{2}\right)^3 \cdot 3\cos^2\frac{t}{2}} \le$$

$$4ab\sqrt{\frac{1}{3}\left(\frac{3\sin^2\frac{t}{2} + 3\cos^2\frac{t}{2}}{4}\right)^4} = 4ab\sqrt{\frac{1}{3}\left(\frac{3}{4}\right)^4} = 4ab\cdot\frac{3\sqrt{3}}{16} = \frac{3\sqrt{3}}{4}ab.$$

Note that equality occurs iff $\sin^2\frac{t}{2} = 3\cos^2\frac{t}{2} \iff \tan\frac{t}{2} = \sqrt{3}$ (because $\tan\frac{t}{2} > 0$ for $t \in (0,\pi)$) that is iff $t = \frac{2\pi}{3} \iff x = \frac{a\sqrt{3}}{2}, y = -\frac{b}{2}$. Then $\max[ABC] = \frac{3\sqrt{3}}{4}ab$ and attained if $A(0,b), B\left(\frac{a\sqrt{3}}{2}, -\frac{b}{2}\right), C\left(-\frac{a\sqrt{3}}{2}, -\frac{b}{2}\right)$.