Math Olympiads Training Problems

Arkady M.Alt

04.24.2016

Abstract

This book is a translated into English extended and significantly added version of author's brochures "Guidelines for teachers of mathematics to prepare students for mathematical competitions" published at 1988 in Odessa.

Preface

This book is a translation into English of my brochures "guidelines for teachers of mathematics to prepare students for mathematical competitions" published 1988 year in Odessa.

More precisely it is corrected and significantly added version of this brochure. In comparison with the first original edition with solutions only to 20 problems from 112 problems represented there this new edition significantly replenished with new problems (around 180 problems).

And now all problems are accompanied by solutions which at different times done by the author of this book (sometimes multivariants and with the analysis and generalizations). Also, unlike the previous edition, all problems are grouped into the corresponding sections of mathematics.

Part I

Methodology Introduction

It makes no sense to repeat what has already been said about the usefulness and expediency of mathematical olympiads of different levels.Therefore, let us dwell on the issues that naturally arise in connection with the Olympiads, in particular, with olympiads of high level, -issues of preparation to Mathematical Competitions

The main question: Is it necessary such preparation?

It's not a secret that students who are able to solve the problems offered at these Olympiads, sufficiently gifted mathematically, have more advanced mathematical techniques and a number

of useful qualities, including the ability to self-organize and independent work.That is, and so good.?

But there is a fact of very serious preparation by level and by time, for participation in international mathematical Olympiads. Is known significant advantages of participants in the Olympiads, students of schools and classes, in which mathematics is taught in a larger volume and with greater depth.

Finally, the more capable a student is, the more important and difficult is to ensure the growing process of improving and systematizing mathematical education, which should include not only the knowledge of concrete facts, but what is more important, the ways of their formation (with the need to include their proofs), intensive practical work with solving non-standard and nonaddressed problems, that is everything that forming a culture of mathematical thinking.

(Culture of mathematical thinking:

-Discipline of thinking, algorithmic thinking, observation, ability to analysis, generalizations, the ability to build mathematical models, to choose a convenient language description of the problem situation .... (the list can be continued)).

The existing system of teaching mathematics in no way contributes to readiness of the student to solve unconventional, nonstandard problems of the Olympiad character. If all this happens, it is not thanks to this system, but contrary to it.

The main reason is that the goal of traditional school education is a certain an admissible minimum of knowledge, limited by the amount of hours, the program, its quantitative and qualitative composition and certainly the teaching methodology based mainly on the memorization of facts and means for execution of algorithmized instructions aimed at solving exclusively typical problems.

If within the framework of this system the student faithfully complies with all the requirements, and limited by this, then his success isn't sufficiently guaranteed. But this is not the main thing.

The main thing is that the creative attitude to mathematics will be hopelessly lost. And if this does not happen in some cases, it is only thanks to the personality of the student and the personality of the teacher that have fallen in the state of resonance.

It's no secret that the assurances of the organizers of the Olympiads that problems do not go beyond of school curriculum to put it mildly, distort the real state of things.

That is, formally they do not sin against the truth, at least so, how much, say, as the editor of the book, which write in the annotation, that for its reading not necessary to have no preliminary information, except for the developed mathematical thinking.

But the latter is already a result of a preparation of very long and intencive and varied, the result of systematic training aimed at developing thinking nonstandard, but logically disciplined.

The essence of this statement will become clear after the complete list of what is know the ordinary student (a student which is in full compliance with the program).

Even in class programs with in-depth study of mathematics, much of this in the following list is missing.

So, what does not know (or know insufficiently) a ordinary school student :

1. Algebraic and analytic technic.

2. Method of mathematical induction at the level of well-developed technique use it in different and, preferably, non-standard situations;

3. The theory of divisibility is, in a volume different from residual, vague representations of a high-school student about knowledge, which was casually received in middle school

4. The algebra of polynomials including the theory of divisibility of polynomials

5. Basic classic inequalities and their applications.

6. Integer and fractional parts. Properties and applications.

7. Technique of solving systems of inequalities in integer numbers and effective representation of integer multidimention domains;

8. Technique of summation, including summation by multidimention domains.

9. Sequences -different ways of their definition (including recursive definition and generating functions) and elementary methods of solving certain classes of recurrence relations and their applications in the theory of divisibility, summation, combinatorics and so on.

10. The Dirichlet principle.

11. Method of invariants.

12. Techniques of elementary (without derivatives) solving extremal problems, especially with many variables.

13. Solving equations in two or more unknowns in integers and especially in non-negative integers.

14. Sequence analysis (boundedness, monotonicity, limit theory, including an theoretical and practical basis, and basic limits). .

To this list it is necessary to add the lack of the ability to solve non-standard, nonaddressed problems. An unconventional, unexpected problem should be classified, understood, reformulated,

simplified, immersed in a more general problem, or treated by special cases and identified the main theoretical tools needed to solve it. The usual work of a student is simple. Here is a chapter, here is a problem to this chapter. Search, recognition work is minimal and the emphasis is exclusively on the robustness of standard algorithms, that is, the minimum problematic level which is the only truly developing thinking factor.

And if thinking does not develop, then it degrades and even ideal diligence can not be a compensation for this loss, accompanying such approach to math education. Thus, a consequtive change of topics, not backed by "no-address" problems does not allow you to achieve the desired effect.

That is, it is necessary that in each topic there are problems that can be solved by using some a previously unknown combination of formally known theoretical propositions from the preceding material.

A participant in the Olympiad needs some psychological qualities that also require training and preparation, either special or spontaneous, accompanying the solution of non-standard problemss in conditions of limited time.

This ability to quickly and deeply focus on a specific problem, quickly relax and switch to another task from any previous emotional state depending on luck or failure.

Required sufficiently rich associative thinking and trained memory, allowing to carry out the associative search for necessary means to solve the problem.

And most importantly, to learn to "misunderstand", that is, to face a problem in which there is nothing to grab on, there is no (at first impression) ready-made approaches to its solution, calmly analyze it to look for something familiar and similar to what you know, consider special cases (reduction), generalize(induction), investigate the problem in limiting cases, introduce additional conditions that simplify the situation, accumulate experimental material.

For a mathematician, a difficult problems is to take height, to overcome not only intellectual barrier but also complexes, fears. Thus, are important the methodical settings of the type: "How to solve the problem?"; psychological attitudes: reaction to the shock of "misunderstanding," the creation of comfortable zones, the ability to relax, adjust to the problems, focus, quickly and deeply dive into it, that is requirements for the student's psychological status.

But the psychological and methodological qualities can only be developed by a large amount of work to solve non-standard problems, with the subsequent analysis of methodological, psychological and

especially technical and ideological aspects, with the formation of generalizing settings, which is also the goal of special training for students.

It often happens that children who are capable of creative work are not able to work in a sporting situation, which of course affects their "sports" results, but does not detract from their ability to mathematical creativity, which is by essence isn't a sports match.

However, as in other areas of human activity, people often bring sports excitement in mathematic, turning it into a competition of minds. By itself it is not a negative quality, but rather useful, developing

motive, under condition that the mathematics by itself is not reducible to one more kind of sports competition.

It should not be forgotten that the Math Olympiad is not an aim in itself, but a training ground on which many qualities necessary for the future researcher are being perfected, such as perseverance,

will, technique, knowledge, skills, reaction to practical situations, thinking.

The list of problems given in this book does not in any way pretend to be complete, but it is quite representative for such sections of school mathematics as arithmetic, algebra and analysis.

The absence of geometric problems proper is caused by the desire to restore the balance in the evidence base to school mathematics.

Traditionally, the concept of proof, the methods of proof, the level of rigor, the axiomatic approach - what we call abstract thinking is basically formed in the course of geometry, that is, a region

closer to sensory perception than algebra.

Arithmetic turns out to abandoned wasteland, somewhere in the backyard of mathematical construction, and algebra is reduced to a set of formulas and rules that need to be remembered and applied.

At the same time, an insignificant part of students informally accepts and understands the level of rigor and evidence in geometry and is able to transfer the acquired thinking technique independently

to other mathematical areas.

For the others, the geometry - not motivated and it is unclear for what sins the punishment by jesuitically sophisticated logic and for some reason mostly "from the opposite", resulting in a

false and unimaginable premise to the almost illusive stunt of drawing a black rabbit from a black hat, "what was required to prove ".

The loss in the geometry of its naturalness, the departure from the exposition of it in the school at the level of Euclid, Kiselev, Kokseter did not bring desirable effect in the plane of it's modernization, rigor or deep understanding, since geometry is not the object on which the axiomatic method, usually accompanied with extreme formalism don't bring true efficiency.

This is particularly true for the introductory courses specific to the school. And although historically, it was in geometry, the axiomatic method showed us its methodological power, geometry

is not at all the only reason for its primary demonstration.

Speaking of this, I in no way deny the use of the elements of the axiomatic method and rigorous proofs in the exposition of geometry, moreover, I consider their use necessary and beneficial.

I am only expressing the doubt, confirmed by my teaching practice, in the appropriateness and effectiveness of primacy in the use of the axiomatic method and proofs in geometry.

It is not with geometry that one has to start the introduction of mathematical formalism, which comes into conflict with the visibility which inherent to geometry.

Due to the structural wealth of geometry, its axiomatics are voluminous and combinatorially saturated and, although grown by abstraction from sensory images, nevertheless do not live well with them.

Rather, they are poorly compatible with the level of rigor and formalism of thinking, which is the inevitable companion of the axiomatic method.

And the roots of this in the psychology of perception and thinking.

It is known that it is most difficult to prove or disprove the apparently obvious, visual:

"The sun revolves around the Earth," "The Earth is flat," "A straight line that crosses one side of the triangle and does not pass through the vertices of a triangle will cross exactly one of its sides."

Hence the Greeks instead of proof drawing and saying "look!".

History begins with geometry, the school copies history, although it is known that when the path is passed, it is not the shortest and most effective.

At the same time, the Peano axioms for natural numbers, the theorems in the divisibility theory, the axioms of various algebraic structures that are essentially a subset of geometric axiomatics are much simpler (combinatorial complexity), less reducible to sensory images, and therefore their use is methodologically more justified.

[Advanced algebraic base assuming free possession of the symbolic transformative technique, systematic proof of all the theoretical facts that make up the qualitative and the computational basis of what is commonly called school algebra (see items 1,2,3, ... of the list of the above) - this should, in my opinion, precede the rest of the school mathematics, including geometry.]

But in school, these topics are taboo. Suffers from such a one-sidedness all math subjects algebra, and arithmetic, and geometry, that is, all of school mathematics and not only.

Mathematics is one, its means are universal - this is the ideological basis on which mathematics education should be carried out. And methodological one-sidedness is unacceptable.

And, finally, the implantation of mathematical methodology into consciousness should should be implemented by the way which is most motivated psychologically.

The lack of habit of abstract reasoning at the level of the proofs of theorems in arithmetic and algebra, in contrast to the intensive theoretical foundation in geometry, subsequently creates a

considerable obstacle in the ability to find arithmetic (algebraic) means, to dispose of them with the same rigor and thoroughness as is customary in geometry.

Quite often the idea of the non-standard and complexity of the arithmetic problem is related precisely to the absence of a completely elementary and essential sequential theoretical basis related to arithmetic of natural, integer and numbers in general, which forms, in addition to everything, is the foundation of mathematical analysis. That is, the non-standard nature of the problem in such cases is equivalent to non-informedness.

In these cases, the situation becomes ambiguous, because, on the one hand, the lack of specific knowledge-tools requires its spontaneous invention in the conditions of the Olympiad, and it is more complicated than choosing the necessary combination of already known tools and technology for its purposeful use, and on the other hand for an informed student solution of the such problem basically becomes a matter of technique.

Thus, the olympiad (sports) value of such problems is doubtful. This does not, however, diminish their possible educational value. But let us leave that on the conscience of the composers of the Olympiad problems and consider the positive aspect of this situation, which consists in motivation of the student in additional technical and theoretical equipment, which ultimately brings him to a higher level, and allows expanding the problem area, then there is more complex problems, the solution of which already depends entirely from recognizing ability of the participant of the Olympiad and his ingenuity in the use of already known means. In this way, both the stimulation of mathematical education and the escalation of thinking take place.

In the author's opinion, the problems presented in the following sections will convincingly argued that was saying in the introduction.

Remark.

1.Abbreviation **n-Met. Rec.**(**Methodical recommendations**) means that the problem originally has number n in the author's brochures "Guidelines for teachers of mathematics to prepare students for mathematical competitions" published at 1988 in Odessa.

2.Abbreviation **MR** means **Mathematical Reflections** – **AwesomeMath**;

3. Abbreviation **ZK** means **Zadachnik Kvanta;**

4. Abbreviation **SSMJ** means **School Science and Mathematics Association Journal**

5. Also, if problem marked by sign ★ it means that the problem was proposed by author of this book.

Part II

Problems

1 Divisibility.

Problem 1.1 (6-Met.Rec.)

Find all n such that $1\underbrace{44\ldots\ 4}_{n\ times}$ is the perfect square.

Problem 1.2 (8-Met.Rec.)

Prove that number $385^{1980}+18^{1980}$ isn't a perfect square.

Problem 1.3 (9-Met.Rec.)

Let $f(x)=x^3-x+1$. Prove that for any natural a numbers $a,\ f(a)\ ,\ f(f(a))\ ,...,$are pairwise coprime.

Problem 1.4(23-Met.Rec.)

Find the largest natural x such that $4^{27}+4^{1000}+4^x$ is a perfect square.

Problem 1.5(24-Met.Rec.)

Prove that 5^n-4^n for any natural $n>2$ isn't perfect square.

Or,

Prove that set $\{5^n-4^n \mid n>2\}$ is free from squares.

Problem 1.6(25-Met.Rec.)

a) Prove that set $\{2^n+4^n \mid n\in\mathbb{N}\}$ is free from squares;

b) Find all non negative integer n and m for which 2^n+4^m is perfect square.

Problem 1.7(26-Met.Rec.)

Find all $n\in\mathbb{N}$ such that 3^n+55 is a perfect square.

Problem 1.8 (27-Met. Rec.)
Prove that the following number is composite for any natural n:
a) $a_n := 3^{2^{4n+1}} + 2$;
b) $b_n := 2^{3^{4n+1}} + 3$;
c) $c_n := 2^{3^{4n+1}} + 5$.

Problem 1.9(28-Met. Rec.)
Prove that $5^n - 1$ isn't divisible by $4^n - 1$ for any $n \in \mathbb{N}$.

Problem 1.10(297-Met. Rec.)
Let a, b, c, d be natural numbers such that $ab = cd$. Prove that for any natural n
number $a^{2n} + b^{2n} + c^{2n} + d^{2n}$ is composite.

Problem 1-11(30-Met. Rec.)
Prove that $5^{3^{4m}} - 2^{2^{4n+2}}$ is divisible by 11 for any natural m, n.

Problem 1.12(32-Met. Rec.)
Is there a number whose square is equal to the sum of the squares of 1000 consecutive
integers?

Problem 1.13(33-Met. Rec.)
Let n be natural number such that $2n + 1$ and $3n + 1$ are perfect squares.
Prove that n is divisible by 40.

Problem 1.14(34-Met. Rec.)
Is it possible that sum of digits of a natural number which is a perfect square be equal 1985?

Problem 1.15
Find all non negative integer n and m for which $2^n + 4^m$ is perfect square.

Problem1.16(37-Met. Rec.)
Prove that:
a) $n!$ isn't divisible by 2^n.
b) $ord_p\left(\left(\left(p-1\right)n\right)!\right) \leq n + ord_p\left(n!\right)$.
c) $(n!)! \geq \left(((n-1)!)!\right)^n$
d) $ord_p\left(\dfrac{(pn)!}{n!}\right) = n$.
e) $(n!)! \geq \left((n-1)!\right)^{n!}$.

Problem 1.17(38-Met. Rec.)
Prove that:

a) $(n!)! \vdots (n!)^{(n-1)!}$;

b) $(n!)! \vdots ((n-1)!)!^n$;

c) $(n^n)! \vdots n!^{n^{n-1}}$;

d) $(n^2)! \vdots (n!)^n$;

e) $(n^{m+k})! \vdots (n^m)!^{n^k}$;

f) $(n \cdot m)! \vdots (n!)^m$;

g) $\dfrac{(2n)!}{n!\,(n+1)!}$ is integer.

h) $(n+1)(n+2)\ldots(n+k) \vdots k!$ for any $n, k \in \mathbb{N}$.

Problem 1.18(41-Met. Rec.)
Find all natural number n such that remainder from division $S_n = 1 + 2 + \ldots + n$ by 5
equal 1.

Problem 1.19(123-Met. Rec.)
Show that the next integer above $\left(\sqrt{3}+1\right)^{2n}$ is divisible by 2^{n+1},i.e. $\left\lceil\left(\sqrt{3}+1\right)^{2n}\right\rceil \vdots 2^{n+1}$.Show that there are infinitely many $n \in \mathbb{N}$ for which $\left\lceil\left(\sqrt{3}+1\right)^{2n}\right\rceil$ not divisible by 2^{n+2}.

2 Diophantine equation.

Problem 2.1(22-Met. Rec.)
Find all integer x such that $\dfrac{3x - \sqrt{9x^2 + 160x + 800}}{16}$ is integer.

Problem 2.2(35-Met. Rec.)
Prove that equation $x^2 - 2xy = 1978$ have no sulutions in integers.

Problem 2.3(47-Met. Rec.)
Prove that if numbers $n, m \in \mathbb{N}$ satisfy to equality $2m^2 + m = 3n^2 + n$ then numbers
$m - n, 2m + 2n + 1, 3m + 3n + 1$ are perfect squares.

Problem 2.4(42-Met. Rec.)
Find all integer solutions of equation $x^3 - 2y^3 - 4z^3 = 0$ (excluding trivial $x = y = z = 0$).

Problem 2.5(43-Met. Rec.)
How many natural solutions have equation $2x^3 + y^5 = z^7$?

Problem 2.6(44-Met. Rec.)
Prove that equation $x^3+y^3+z^3+t^3=u^4-v^4$ has infinitely many solutions in
natural x, y, z, t, u, v.

Problem 2.7(45-Met. Rec.)
How many natural solutions has equation $x^4+y^6+z^{12}=t^4$?

Problem 2.8(46-Met. Rec.)
Prove that for any given integer t the following equations have no integer solutions:
a) $x^3+y^3=9t\pm 4$;
b) $x^3+y^3=9t\pm 3$;
c) $x^3+y^3+z^3=9t\pm 4$;
d) $x^3+117y^3=5$.

Problem 2.9(50-Met. Rec.)
Let a be integer number such that $3a=x^2+2y^2$ for some integer numbers x, y.
Prove that number a can be represented in the same form , that is there is
integers u, v that $a=u^2+2v^2$.

Problem 2.10(40-Met. Rec.)
Find conditions for irreducible fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ that provide silvability of equation
$y=x^2+\dfrac{a}{b}x+\dfrac{c}{d}$ in integer x, y.(that parabola contain at least one (then infinitely many)
points $M(x, y)$ with integer x, y.

★ **Problem 2.11(3932, CRUX)**
Prove that for any natural numbers x, y satisfying equation $x^2-14xy+y^2-4x=0$
holds $\gcd^2(x, y)=4x$.

Problem 2.12(54-Met. Rec.)
The store has a sealant in boxes of 16lb, 17lb, 21lb. How some organization can get without
opening boxes 185 lb of sealant and so, that the number of boxes was the smallest?

Problem 2.13(55-Met. Rec.)
Find the number of non-negative integer solutions of equation $5x+2y+z=10n$ in term of

given natural n.

3 Integer and fractional parts.

Problem 3.1 (56-Met. Rec.)
Find $\left[\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3\right]$.

Problem 3.2 (57-Met. Rec.)
Simplify
a) $\left[\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\right]$;
b) $\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right]$.

Problem 3.3 (59-Met. Rec.)
Solve equation $\{x\}+\left\{\dfrac{1}{x}\right\}=1, x\in\mathbb{R}$.

Problem 3.4 (60-Met. Rec.)
Prove equality $\sum\limits_{a=2}^{n}[\log_a n]=\sum\limits_{b=2}^{n}[\sqrt[b]{n}]$.

★Problem 3.5 (3095, CRUX)
Let a, b, c, p, and q be natural numbers. Using $\lfloor x\rfloor$ to denote the integer part of x, prove that

$$\min\left\{a,\left\lfloor\frac{c+pb}{q}\right\rfloor\right\}\leq\left\lfloor\frac{c+p\,(a+b)}{p+q}\right\rfloor.$$

Problem 3.6 (10-Met. Rec.)
Prove that:
a) For any $n\in\mathbb{N}$ holds inequality $\left\{n\sqrt{2}\right\}>\dfrac{1}{2n\sqrt{2}}$;
b) For any $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $\left\{n\sqrt{2}\right\}<\dfrac{1+\varepsilon}{2n\sqrt{2}}$.

Problem 3.7 (11-Met. Rec.)
Let $n\in\mathbb{N}$ isn't forth degree of natural number. Then

$$\{\sqrt[4]{n}\}>\frac{1}{4n^{3/4}}.$$

★Problem 3.8 (J289,MR)
For any real $a\in[0,1)$ prove the following identity

$$\left[a\left(1+\left[\frac{1}{1-a}\right]\right)\right]+1=\left[\frac{1}{1-a}\right].$$

Problem 3.9 (118-Met. Rec.)
For arbitrary natural $m\geq 2$ prove that $\left\lfloor\left(m+\sqrt{m^2-1}\right)^n\right\rfloor$ is odd number for any natural n.

★ Poblem 3.10 (W16, J.Wildt IMO 2017)
For given natural $n > 1$ find number of elements in image of function
$$k \mapsto \left[\frac{k^2}{n}\right] : \{1, 2, ..., n\} \longrightarrow \mathbb{N} \cup \{0\}.$$

4 Equations, systems of equations.

★Problem 4.1(90-Met. Rec.)(Generalization of M703* Kvant)
Solve the system of equations.
$$\begin{cases} (q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z) \\ xy + yz + zx = 1 \end{cases}$$
where p, q, r are positive real numbers.

Problem 4.2 (91-Met. Rec.)
Solve the system of equations
$$\begin{cases} 2x + x^2 y = y \\ 2y^2 + y^2 z = z \\ 2z^2 + z^2 x = x \end{cases}.$$

Problem 4.3 (92-Met. Rec.)
Solve the system of equations:
$$\begin{cases} x - y = \sin x \\ y - z = \sin y \\ z - x = \sin z \end{cases}.$$

Problem 4.4 (93-Met. Rec.)
Solve the system of equations:
$$\begin{cases} x_1 + x_2 + ... + x_n = 1 \\ x_1^2 + x_2^2 + ... + x_n^2 = \dfrac{1}{n} \end{cases}.$$

Problem 4.5 (94-Met. Rec.)
Solve the system of equations:

a) $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y + az = 1 + a \end{cases}, a \geq \dfrac{1}{2};$

b) $\begin{cases} x + y + z = a \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{a} \end{cases}, a \neq 0.$

Problem 4.6 (95-Met. Rec.)
Given that
$$\begin{cases} x + y + z = 2 \\ xy + yz + zx = 1 \end{cases}.$$
Prove that $x, y, z \in [0, 4/3]$.

Problem 4.7(96-Met. Rec.)
Solve the system of equations:
$$\begin{cases} 2(\cos x - \cos y) = \cos 2x \cos y \\ 2(\cos y - \cos z) = \cos 2y \cos z \\ 2(\cos z - \cos x) = \cos 2z \cos x \end{cases}.$$

5 Functional equations and inequalities

Problem 5.1 (97-Met. Rec.)

Find all functions defined on $\mathbb{R}$ such that:

a) $f\left(x^2\right) - (\ f(x))^2 \geq 1/4$ and $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$;

b) $f(x) \leq x$ for any $x \in \mathbb{R}$ and $f(x+y) \leq f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

Problem 5.2 (99-(Met. Rec.)

Function $f(x)$ defined on $[0,1]$ and satisfies to equation $f(x + f(x)) = f(x)$ for any $x \in [0,1]$ .Prove that $f(x) = 0$ for all $x \in [0,1]$.

Problem 5.3 (100-Met. Rec.)

Find all continuous on $\mathbb{R}$ functions f suth that $f(x) f(y) - xy = f(x) + f(y) - 1$ holds for any $x, y \in \mathbb{R}$.

Problem 5.4 (101-Met. Rec.)

Let $n \in \mathbb{N} \setminus \{1\}$. Find all defined on $\mathbb{R}$ functions f such that $nf(nx) = f(x) + nx$ for any $x \in \mathbb{R}$ and f is continuous in $x = 0$.

Problem 5.5 (14-Met. Rec.)

Prove that there is no function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous on $\mathbb{R}$ such that

$$f(x+1)(f(x)+1) + 1 = 0.$$

Problem 5.6 (15-Met. Rec.)

For any given $n \in \mathbb{N}$ find all functions $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that

$$f(m+k) = f(mk - n),\ m, k \in \mathbb{N} \text{ and } mk > n.$$

★Problem 5.7 (U182,MR)

Find all continuous on $[0,1]$ functions f such that $f(x) = c$, if $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = f(2x-1)$ if $x \in \left(\frac{1}{2}, 1\right]$,where c is given constant.

6 Recurrences.

Problem 6.1 (4-Met. Rec.)

Let p is some natural number. Prove, that exist infinitely many pairs (x, y) of natural numbers such, that $\dfrac{x^2+p}{y}$ and $\dfrac{y^2+p}{x}$ are integer numbers.

Problem 6.2 (5-Met. Rec.)

Let sequence is defined recursively as follow:

$$a_{n+3} = \frac{a_{n+1}a_{n+2}+5}{a_n}, n \in \mathbb{N} \text{ and } a_1 = a_2 = 1, a_3 = 2.$$

Prove that all terms of this sequence are integer numbers.

Problem 6.3 (16-Met. Rec.Problem 5, Czechoslovakia, MO 1986)

Sequence of integer numbers $a_1, a_2, ..., a_n, ...$ defined as follows:

$a_1 = 1, a_{n+2} = 2a_{n+1} - a_n + 2, n \in \mathbb{N}$.

Prove that for any $n \in \mathbb{N}$ there is such $m \in \mathbb{N}$ that $a_n a_{n+1} = a_m$.

Problem 6.4 (17-Met. Rec.)

Prove that if sequence $(a_n)_{n\geq 1}$ satisfy to recurrence $a_{n+2} = a_{n+1}^2 - a_n, n \in \mathbb{N}$ with initial

conditions $a_1 = 39, a_2 = 45$ then infinitely many terms of this sequence is divisible

by 1986.

Problem 6.5* (31-Met. Rec.)

Given a quad of integer numbers (a, b, c, d) such that at least two of them are different.

Starting from this quad we create new quad $(a_1, b_1, c_1, d_1) = (a-b, b-c, c-d, d-a)$.

By the same way from quad (a_1, b_1, c_1, d_1) we obtain quad (a_2, b_2, c_2, d_2) and so on...

Prove that at least one from the numbers $a_{100}, b_{100}, c_{100}, d_{100}$ bigger than 10^9.

Problem 6.6* (19-Met. Rec)

Let $a > 1$ is natural number. Sequence $a_1, a_2, ..., a_n, ...$The sequence is defined recursively

$$\begin{cases} a_1 = a \\ a_n = a^n - \sum\limits_{t|n,\ t<n} a_t \end{cases}.$$

Prove that $a_n \vdots n$ for any $n \in \mathbb{N}$ (a_n divisible by n for any $n \in \mathbb{N}$).

Problem 6.7* (G.Demirov, Matematika 1989,No.7 ,p.34, Bolgaria)

Let sequence (a_n) defined by the recurrence

$a_{n+2} = a_{n+1}a_n - 2(a_{n+1} + a_n) - a_{n-1} + 8, n \in \mathbb{N}$ with initial conditions

$a_0 = 4, a_1 = a_2 = (a^2-2)^2$, where $a \geq 2$.

Prove that for any $n \in \mathbb{N}$ expression $2 + \sqrt{a_n}$ is a square of some polinomial of a.

Problem 6.8

Find general term of the sequence:

a) $a_{n+1} = \frac{1}{27}\left(8 + 3a_n + 8\sqrt{1+3a_n}\right), a_1 = \frac{8}{3}$;

b) $a_{n+1} = \frac{1}{16}\left(1 + 4a_n + \sqrt{1+24a_n}\right), a_1 = 1.$

Problem 6.9*

Let sequence (a_n) be defined by equation $\left(\sqrt{2}-1\right)^n = \sqrt{a_n+1} - \sqrt{a_n}$.

a) Find recursive definition for (a_n) and prove that a_n is integer for all natural n;

b) Let $t_n := \sqrt{2a_n(a_n+1)}$. Find recursive definition for (t_n) and prove that t_n is integer for all natural n.

Problem 6.10.(Proposed by S. Harlampiev, Matematika 1989, No.2, p,43, Bolgaria)

Let sequence defined by recurrence

$$a_{n+2} = \frac{2a_{n+1} - 3a_{n+1}a_n + 17a_n - 16}{3a_{n+1} - 4a_{n+1}a_n + 18a_n - 17}, n \in \mathbb{N} \cup \{0\}$$

with initial conditions $a_0 = a_1 = 2$.

Prove that a_n for any $n \in \mathbb{N} \cup \{0\}$ can be represented in the form $1 + \dfrac{1}{m^2}$ where $m \in \mathbb{N}$.

Problem 6.11*. (Proposed by Bulgaria for 1988 IMO)

Let $a_0 = 0, a_1 = 1, a_{n+1} = 2a_n + a_{n-1}, n \in \mathbb{N}$. Prove that $a_n \vdots 2^k \iff n \vdots 2^k$.

7 Behavior(analysis) of sequences

Problem 7.1 (104-Met.Rec)

For natural $n \geq 3$ let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 = a_n = 0$ and $a_{k-1} + a_{k+1} \leq 2a_k, k = 2, ..., n-1$. Prove that $a_k \geq 0, i = 1, 2, ..., n$.

Problem.7.2 (105-Met. Rec.)

a) Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n}, n \in \mathbb{N}$. Prove that $14 < a_{100} < 18$.

Find lower and upper bounds for a_n.**(Problem 7 from all Soviet Union Math Olympiad,1968)**

b) Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n^2}, n \in \mathbb{N}$.

i.Prove that (a_n) unbounded.

ii. $a_{9000} > 30$;

iii.★ find good (assimptotic) bounds for (a_n).

Problem 7.3 (106-Met. Rec.)

Find all values of a,such that sequence $a_0, a_1, ..., a_n, ...$ defined as follows $a_0 = a, a_{n+1} = 2^n - 3a_n, n \in \mathbb{N} \cup \{0\}$ is increasing sequence.

Problem 7.4 (107-Met. Rec.)

Known that sequence $a_1, a_2, ..., a_n, ...$ satisfy to inequality
$a_{n+1} \leq \left(1 + \dfrac{b}{n}\right) a_n - 1, n \in \mathbb{N},$ where $b \in [0, 1)$.
Prove that there is n_0 such that $a_{n_0} < 0$.

Problem 7.5* (109-Met.Rec.) (Team Selection Test, Singapur).
Let $n \in \mathbb{N}$, $a_0 = \dfrac{1}{2}$ and $a_{k+1} = a_k + \dfrac{a_k^2}{n}, k \in \mathbb{N}$. Prove that
$$1 - \frac{1}{n} < a_n < 1.$$

Problem 7.6 (110-Met. Rec.)
Find $\lim\limits_{n\to\infty} \left\{ \left(2 + \sqrt{3}\right)^n \right\}$.

Problem 7.7 (111 -Met. Rec.)
a) Let sequence (x_n) satisfy to recurrence $x_{n+1} = x_n (1 - x_n)$, $n \in \mathbb{N} \cup \{0\}$ and $x_0 \in (0, 1)$. Prove that $\lim\limits_{n\to\infty} n x_n = 1$;

b) Let sequence (x_n) satisfy to recurrence $x_{n+1} = x_n^2 - x_n + 1$ and $x_1 = a > 1$.

i. Find $\sum\limits_{n=1}^{\infty} \dfrac{1}{x_n}$;

ii. Find $\left\lfloor \dfrac{x_{n+1}}{x_1 x_2 ... x_n} \right\rfloor$.

c) Let sequence (x_n) satisfy to recurrence $x_n = 0.5 x_{n-1}^2 - 1, n \in \mathbb{N}$ with initial
condition $x_0 = \dfrac{1}{3}$.
Find $\lim\limits_{n\to\infty} x_n$.

Problem 7.8 (112-Met. Rec.)
Find $\lim\limits_{n\to\infty} x_n$ where $x_0 = 1/3, x_{n+1} = 0.5 x_n^2 - 1, n \in \mathbb{N} \cup \{0\}$.

Problem 7.9* Let $a_1 = \dfrac{1}{2}$ and $a_{n+1} = a_n - n a_n^2, n \in \mathbb{N}$.

a) Prove that $a_1 + a_2 + ... + a_n < \dfrac{3}{2}$ for all $n \in \mathbb{N}$.

b*) Find "good" bounds for a_n,i.e. such two well calculating function $l(n)$ and $u(n)$
such that $l(n) \leq a_n \leq u(n)$ for all n greater then some n_0 and $\lim\limits_{n\to\infty} \dfrac{a_n}{u(n)} = \lim\limits_{n\to\infty} \dfrac{a_n}{l(n)} = 1$

(This equivalent to $\lim\limits_{n\to\infty} \dfrac{a_n}{l(n)} = \lim\limits_{n\to\infty} \dfrac{u(n)}{l(n)} = 1$ or to $\lim\limits_{n\to\infty} \dfrac{a_n}{u(n)} = \lim\limits_{n\to\infty} \dfrac{l(n)}{u(n)} = 1$);

We call two function $l(n)$ and $u(n)$ asymtotically equal and write it $l(n) \sim u(n)$

if $\lim\limits_{n\to\infty}\dfrac{l(n)}{u(n)}=1$. Thus function $l(n)$ and $u(n)$ is good bound iff $l(n)\sim u(n)$ and

$l(n)\le a_n\le u(n)$.

c) Determine asymptotic behavior of a_n ,i.e. find function asymptotically equal to a_n.

(or more simple question: Find $\lim\limits_{n\to\infty} n^2 a_n$).

Problem 7.10

Let sequence (a_n) satisfy to recurrence $a_{n+1}=\dfrac{a_n^2-2}{2}, n\in\mathbb{N}$.Prove that:

i. If $a_1=1$ then (a_n) is bounded;

ii. If $a_1=3$ then (a_n) is unbounded.

Problem 7.11

Let sequence (a_n) defined by $a_1=1, a_{n+1}=\dfrac{3}{4}a_n+\dfrac{1}{a_n}, n\in\mathbb{N}$. Prove that:

i. (a_n) is bounded;

ii. Prove that $|a_n-2|<\left(\dfrac{2}{3}\right)^n, n\in\mathbb{N}$.

Generalization: $a_{n+1}=pa_n+\dfrac{1}{a_n}$, $n\in\mathbb{N}$ for any given $p\in(0,1)$.

Problem 7.12 (Bar-Ilan University math. olympiad, Israel).

Let $a_1=1, a_{n+1}=1+\dfrac{1}{a_n}, n\in\mathbb{N}$. Prove that there is real number b which for all $n\in\mathbb{N}$ satisfy inequality $a_{2n-1}<b<a_{2n}$.

Problem 7.13

Let $a_0=1$and $a_{n+1}=\dfrac{a_n}{2}+\dfrac{1}{a_n}$ for $n=0,1,2,.....$Prove that $\dfrac{2}{\sqrt{a_n^2-2}}$ is an

integer for every natural n.

Problem 7.14

Let $a_0=1$and $a_{n+1}=\dfrac{a_n}{2}+\dfrac{1}{a_n}$ for $n=0,1,2,.....$Prove that $\dfrac{2}{\sqrt{a_n^2-2}}$ is an

integer for every natural n.

Problem 7.15(All Israel Math. Olympiad in Hayfa)

Given m distinct, non-zero real numbers $a_1,a_2,...,a_m, m>1$.Let for any natural $r\ge 1$

$A_r=a_1^r+a_2^r+..+a_m^r$. Prove that for odd m inequality $A_r\neq 0$ holds for all r up to finite

set of values r.

Problem 7.16*(#7,9-th grade,18-th All Soviet Union Math Olympiad,1984,,

Proposed by Agahanov N.H.)
Let $x_1 = 1, x_2 = -1$ and $x_{n+2} = x_{n+1}^2 - \frac{x_n}{2}, n \in \mathbb{N}$.
Find $\lim\limits_{n\to\infty} x_n$.

Problem 7.17
Given sequence of positive numbers (a_n) such that $a_{n+1} \le a_n (1 - a_n)$.
Prove that sequence (na_n) is bounded.

Problem 7.18 (BAMO-2000)
Given sequence (a_n) such that $a_1 > 0$ and $a_n^2 \le a_n - a_{n+1}, n \in \mathbb{N}$.
Prove that $a_n < \frac{1}{n}$ for all natural $n \ge 2$.

★Problem 7.19 (SSMJ 5281)
For sequence $\{a_n\}_{n\ge1}$ defined recursively by $a_{n+1} = \frac{a_n}{1 + a_n^p}$ for $n \in \mathbb{N}$, $a_1 = a > 0$,
determine all positive real p for which series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

Problem 7.20
Given $a_1 = 5, a_{n+1} = a_n^2 - 2, n \in \mathbb{N}$.
a) Find $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_1 a_2 ... a_n}$;
b) Find $\lim\limits_{n\to\infty} \left(\frac{1}{a_1} + \frac{1}{a_1 a_2} + ... + \frac{1}{a_1 a_2 ... a_n} \right)$.

Problem 7.21*
Let $a_1 = a$,where $a > 0, a_{n+1} = \frac{a_n}{1 + \sqrt{a_n}}$, $n \in \mathbb{N}$.
a) Prove that sum $S_n = a_1 + a_1 + ... + a_n$ is bounded ;
b*) ★ Find "good" bounds for a_n if $a_1 = 9$.(Or find asymptotic representation for a_n)
c) Find the $\lim\limits_{n\to\infty} n^2 a_n$.

★ Problem 7.22 (One asymptotic behavior) (S183)

Let sequence (p_n) satisfied to recurrence $p_n = p_{n-1} - \frac{p_{n-1}^2}{2}, n = 1, 2, ...$ and $p_0 \in (0, 1)$.

Prove that $\frac{2}{n + \sqrt{n} + p + 1} < p_n < \frac{2}{n + p}, n \in \mathbb{N},$ where $p := \frac{2}{p_0}$.

8 Inequalities and max,min problems.

Comparison of numerical expressions.

Problem 8.1 (81-Met. Rec.)
Determine which number is greater (here $\vee$ is sign of ineqeality $<$ or $>$ in unsettled state).

a) $31^{11} \vee 17^{14}$;
b) $127^{23} \vee 513^{18}$;
c) $53^{36} \vee 36^{53}$;
d) $\tan 34^\circ \vee \dfrac{2}{3}$;
e) $\sin 1 \vee \log_3 \sqrt{2}$;
f) $\log_{(n-1)} n \vee \log_n (n+1)$;
h) $100^{300} \vee 300!$;
g) $(n!)^2 \vee n^n$;
i) $\sqrt{2+\sqrt{3+\sqrt{2+...}}} \vee \sqrt{3+\sqrt{2+\sqrt{3+...}}}$ (n roots in each expression).
For any natural n compare two numbers $a_n = \sqrt{2+\sqrt{3+\sqrt{2+...}}}$ and $b_n = \sqrt{3+\sqrt{2+\sqrt{3+...}}}$ (each use n square root simbols).What is greater?

Proving inequalities

Problem 8.2 (Inequality with absolute value)
Let a, b, c be real numbers such that $a+b+c=0$.Prove that
$|a \cdot b \cdot c| \le \dfrac{1}{4} \max\left\{|a|^3, |b|^3, |c|^3\right\}$.

Problem 8.3 (69-Met. Rec.)
Let $x, y, z \ge 0$ and $x+y+z \le \dfrac{1}{2}$. Prove that
$(1-x)(1-y)(1-z) \ge \dfrac{1}{2}$.

Problem 8.4 (Problem 6 from 6-th CGMO, 2-nd day,2007).
For nonnegative real numbers a, b, c with $a+b+c=1$, prove that
$$\sqrt{a+\frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \le \sqrt{3}.$$

Problem 8.5 (70-Met. Rec.)
Prove that for any positive real $a_1, a_2, ..., a_n, n \ge 3$ holds inequality
$\sum\limits_{cyc}^{n} \dfrac{a_1 - a_3}{a_2 + a_3} \ge 0$ (Or, $\sum\limits_{i=1}^{n} \dfrac{a_i - a_{i+2}}{a_{i+1} + a_{i+2}} \ge 0$ $(a_{n+1} = a_1, a_{n+2} = a_2)$.

Problem 8.6 (71-Met. Rec.)
For any positive real a, b, c prove inequality
$\dfrac{a^3}{a^2+ab+b^2} + \dfrac{b^3}{b^2+bc+c^2} + \dfrac{c^3}{c^2+ca+a^2} \ge \dfrac{a+b+c}{3}$.

Problem 8.7 (72-Met. Rec.)
Prove that any nonnegative real a, b, c holds inequlity
$a^5 + b^5 + c^5 \ge abc(ab+bc+ca)$.

Problem 8.8 (74-Met. Rec.)

Prove that $\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \leq \sqrt{21}$ if $a,b,c > -\dfrac{1}{4}$ and $a+b+c = 1$.

Problem 8.9 (75-Met. Rec.)

Prove that $(x_1+x_2+...+x_n+1)^2 \geq 4\left(x_1^2+x_2^2+...+x_n^2\right), x_i \in [0,1], i = 1,2,...,n$.

Problem 8.10 (76-Met. Rec.)

Let x,y,z be positive real numbers. Prove that $x^3z+y^3x+z^3y \geq xyz(x+y+z)$.

Problem 8.11 (77-Met. Rec.)(Oral test in MSU)

Solve inequality $\left(\dfrac{x_1y_1+x_2y_1}{x_1y_1+x_1y_2}\right)^{x_1}\left(\dfrac{x_2y_2+x_1y_2}{x_2y_2+x_2y_1}\right)^{x_2} \geq 1$ for real positive x_1, x_2, y_1, y_2.

Problem 8.12 (78-Met. Rec.)

Prove inequality

$$\sqrt{2}+\sqrt{4-2\sqrt{2}}+\sqrt{6-2\sqrt{6}}+...+\sqrt{2n-2\sqrt{n(n-1)}} \geq \sqrt{n(n+1)}.$$

Problem 8.13 (79-Met. Rec.)

Given that $a_1,a_2,...,a_n$ are positive numbers and $a_1+a_2+...+a_n = 1$.Prove that

$$\sum_{k=1}^{n} a_k\sqrt{1-\left(\sum_{i=1}^{k} a_i\right)^2} < \frac{4}{5}.$$

Problem 8.14 (84-Met. Rec.)

Let $a_1,a_2,...,a_n$ be positive real numbers. Prove that

$$a_1a_2+a_2a_3+...+a_{n-1}a_n+a_na_1 \leq \begin{cases} \dfrac{(a_1+a_2+...+a_n)^2}{n}, & \text{if } n=2,3 \\ \dfrac{(a_1+a_2+...+a_n)^2}{4}, & \text{if } n \geq 4 \end{cases}.$$

Problem 8.15 (85-Met. Rec.).Original setting.

Prove that for any numbers $a_1,a_2,...,a_n \in [0,2], n \geq 2$ holds inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|a_i-a_j| \leq n^2.$$

*More difficult variant of the problem:

Find max $\sum_{1\leq i<j\leq n}^{n}|a_i-a_j|$, if $a_1,a_2,...,a_n$ be any real numbers such that $|a_i-a_j| \leq 2, i,j \in \{1,2,...,n\}$.

★ **Problem 8.16 (as modification of S97,MR)**

For any real $x_1,x_2,...,x_n$ such that $x_1+x_2+...+x_n = n$ prove inequality

$$x_1^2x_2^2...x_n^2\left(x_1^2+x_2^2+...+x_n^2\right) \leq n.$$

★Problem 8.17 (W6, J. Wildt IMO, 2014)
Let D_1 be set of strictly decreasing sequences of positive real numbers with first term equal to 1.For any $\mathbf{x}_{\mathbb{N}} := (x_1, x_2, ..., x_n, ...) \in D_1$ prove that
$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \geq \frac{4}{9}$$
and find the sequence for which equality occurs.

★ Problem 8.18 (SSMJ 5345)
Let $a, b > 0$. Prove that for any x, y holds inequality
$$|a \cos x + b \cos y| \leq \sqrt{a^2 + b^2 + 2ab \cos (x + y)}$$
and find when equality occurs.

★Problem 8.19
For any natural n and m prove inequality
$$\left(n^m + n^{m-1} + ... + n + 1\right)^n \geq (m+1)^n (n!)^m .$$

★Problem 8.20
Prove that $(n+1) \cos \dfrac{\pi}{n+1} - n \cos \dfrac{\pi}{n} > 1$ for any natural $n \geq 2$.

Finding maximum,minimum and range.

Problem 8.21 (82-Met. Rec.)
Find the min $\dfrac{-x^2 + 2x - 1}{6x^2 - 7x + 3}$ without using derivative.

Problem 8.22 (83-Met. Rec.)
Let $S(x, y) := \min \left\{ x, \dfrac{1}{y}, y + \dfrac{1}{x} \right\}$ where x, y be positive real numbers. Find $\max\limits_{x,y} S(x, y)$.

★Problem 8.23(58-Met. Rec.).

Find the maximal value of remainder from division of natural number n by natural number a,
where $1 \leq a \leq n$ ($\max\limits_{1 \leq a \leq n} r_a(n), n \in \mathbb{N}$).

★Problem 8.24
Find $\min\limits_{x,y,z} F(x, y, z)$,where $F(x, y, z) = \max \{|\cos x| + |\cos 2y|, \ |\cos y| + |\cos 2z|, \ |\cos z| + |\cos 2x|\}$.

Problem 8.25 (73-Met. Rec.)(M1067, ZK)
Let x, y, z be positive real numbers such that $xy + yz + zx = 1$.
Find the minimal value of expression $\dfrac{x}{1 - x^2} + \dfrac{y}{1 - y^2} + \dfrac{z}{1 - z^2}$.

★Problem 8.26 (SSMJ 5404)**
For any given positive integer $n \geq 3$ find smallest value of product $x_1 x_2 ... x_n$ where $x_1, x_2, ..., x_n > 0$ and $\dfrac{1}{1 + x_1} + \dfrac{1}{1 + x_2} + ... + \dfrac{1}{1 + x_n} = 1$.

9 Invariants.

Problem 9.1 (65-Met. Rec.).

a) An arbitrary fraction $\frac{a}{b}$ may be replaced by one of the fractions $\frac{a-b}{b}, \frac{a+b}{b}, \frac{b}{a}$. Is it possible that after several such transformation starting with fraction 1/2 obtain the fraction 67/91?

b) An arbitrary pair of fraction $\left(\frac{a}{b}, \frac{c}{d}\right)$may be replaced by one the following pairs of fractions

$\left(\frac{a+b}{b}, \frac{c+d}{d}\right), \left(\frac{a-b}{b}, \frac{c-d}{d}\right), \left(\frac{b}{a}, \frac{d}{c}\right)$.

Is it possible that after several such transformation starting with the pair $(1/2, 3/4)$ obtain the
the pair $(5/6, 9/11)$?

c) Given the triple of number $\left(2, \sqrt{2}, 1/\sqrt{2}\right)$ .Allowed any two numbers from current triple (a, b, c)
replace with their sum divided by $\sqrt{2}$ and their difference divided by $\sqrt{2}$. Is it possible after some
numbers of allowed transformations obtain the triple $\left(1, \sqrt{2}, \sqrt{2}-1\right)$.

Problem 9.2 (66-Met. Rec.).

On Rainbow Island living 13 red, 15 green, 17 yellow chameleons. When two chameleons of
one color meet each other then nothing happens, but if they have different color, they both change
the color to the third one. Is it possible that with time all chameleons on island became of one color?

Problem 9.3

In the box are 13 red and 17 white balls. Permitted in any order and
any number of the following operations:
1. Remove from the box one red ball and put it in a box two white balls;
2. Put it in a box one red ball and two white balls;
3. Remove from the box two red balls and put it in a box one white ball;
4. Remove from the box one red ball and two white balls.
Is it possible that after some number of permitted operations to lay in
the box 37 red and 43 white balls?

9.1 Miscellaneous problems.

Problem 10.1 (1-Met. Rec.)

The 8 pupils bring from forest 60 mushrooms. Neither two from them bring mushrooms equally.

Prove that among those pupils has three pupils, whose collect amount of mushrooms not less
than the other five pupils.

Problem 10.2 (2-Met. Rec.)
2000 apples lies in several baskets. Permitted to remove the basket and removing any
number of the apples from baskets.
Prove it's possible to obtain situation that in all baskets that remains numbers of apples
are equal and common number of apples would be not less then 100.

Problem 10.3 (3-Met. Rec.)
Prove that digit of tens in 3^n is even number.

Problem 10.4 (7-Met. Rec.)
Does exist natural number such that first 8 digits after decimal dot of $\sqrt{n}$ are 19851986?

Problem 10.5 (12-Met. Rec.)
Prove that if $2a+3b+6c=0, a \neq 0$ then quadratic equation $ax^2+bx+c=0$ has at least one root on the interval $(0,1)$.

Problem 10.6 (13-Met. Rec.)
Prove that if $a(4a+2b+c)<0$ then $b^2>4ac$.

Problem 10.7 (18-Met. Rec.)
Prove that derivative of function
$$f(x)=\frac{x-1}{x-2}\cdot\frac{x-3}{x-4}\cdot\ldots\cdot\frac{x-2n+1}{x-2n}$$
is negative in all points of domain of $f(x)$.

Problem 10.8 (20-Met. Rec.)
Is it always from the sequence of n^2+1-th numbers $a_1,a_2,...,a_{n^2},a_{n^2+1}$ is possible to
select a monotonous subsequence of lenght $n+1$?

Problem 10.9 (22-Met. Rec.)
Let natural numbers n,m satisfy inequality $\sqrt{7}-\frac{m}{n}>0$. Then holds inequality
$$\sqrt{7}-\frac{m}{n}>\frac{1}{mn}$$
As variant : Find max $\left\{m^2-7n^2 \mid m,n\in\mathbb{N} \text{ and } \frac{m}{n}<\sqrt{7}\right\}$.

Problem 10.10 (39-Met. Rec.)

Rational number represented by irreducible fraction $\dfrac{p}{q}$ belong to interval $\left(\dfrac{6}{13},\dfrac{7}{15}\right)$.

Prove that $q \geq 28$.

★Problem 10.11

Find all one hundred digits numbers such that each of them equal to sum that addends are

all its digits, all pairwise products of its digits and so on,... and at last product of all its digits.

Problem 10.12 (51-Met. Rec.)

Let $P(x)$ be polynomial with integer coefficients. Known that $P(0)$ and $P(1)$ are odd

numbers. Prove that $P(x)$ have no integer roots.

Problem 10.13 (52-Met. Rec.)

Known that value of polynomial $P(x)$ with integer coefficients in three different points equal to 1.

Is it possible that $P(x)$ has integer root?

Problem 10.14 (53-Met. Rec.)

Let $P(x)$ be polynomial with integer coefficients and $P(n) = m$ for some integer $n, m (m \neq 0)$.

Then $P(n + km)$ divisible by m for any natural k.

Problem 10.15 (61-Met. Rec.)

Find the composition $g(x) = \underbrace{f(f(...f(x)...))}_{n-\text{times}}$, where

a) $f(x) = \dfrac{x}{\sqrt{1-x^2}}$;

b) $f(x) = \dfrac{x\sqrt{3}-1}{x+\sqrt{3}}$.

Problem 10.16 (62-Met. Rec.)

Let $F(x) = \dfrac{4^x}{4^x+2}$.Find $F\left(\dfrac{1}{1988}\right) + F\left(\dfrac{2}{1988}\right) + ... + F(1)$.

Problem 10.17(63-Met. Rec.)

Let $f(q)$ the only root of the cubic equation $x^3 + px - q = 0$,where p is given positive real number.

Prove that $f(q)$ is increasing function in $q \in \mathbb{R}$.

Problem 10.18 (64-Met. Rec.)

Let $P(x)$ be a polynomial such that equation $P(x) = x$ have no roots. Is there a root of the equation $P(P(x)) = x$?

Problem 10.19 (67-Met. Rec).

The two rows of boys and girls set (in the first row, all boys, all girls in the second row),

so that against every girl stand the boy that not lower than girl, or differs by the growth

from her not more than 10 cm.

Prove that if children positioned in the each row accordingly their growth then against

each girl will be a boy which again not lower than girl, or differs by the growth

from her not more than 10 cm.

Problem 10.20 (86-Met. Rec.)

Find all values of real parameter b for which system

$$\begin{cases} x \geq (y-b)^2 \\ y \geq (x-b)^2 \end{cases}$$

has only solution.

★Problem 10.21 (CRUX 3090)

Find all non-negative real solutions (x, y, z) to the following system of inequalities:

$$\begin{cases} 2x(3-4y) \geq z^2+1 \\ 2y(3-4z) \geq x^2+1 \\ 2z(3-4x) \geq y^2+1 \end{cases}.$$

★Problem 10.22 (87-Met.Rec.)

Let A_1, A_2, A_3, A_4 be consequtive points on a circle and let a_i is number of rings on the

rod at the point $A_i = 1, 2, 3, 4$. Find the maximal value of $2-$rings chains, that can be

constructed from rings taken by one from any 2 neighboring, staying in cyclic order, rods.

Problem 10.23 (Problem with light bulbs).

n light bulbs together with its switches initially turned off arranged in a row and numbered

from left to right consequtively by numbers from 1 to n.

If you click to the k-th switch than all light bulbs staying on the places numbered

by multiples of k change state (turned off, turned on).

Some person moving from left to right along a row of light bulbs switch clicks each

bulb (once). How many bulbs will light up when he comes to the last light bulb.

★Problem 10.24 (O274, MR4,2013).
Let a, b, c nonnegative integer numbers such that a and b are relatively prime.
How many lattice points belong to domain
$D := \{(x, y) \mid x, y \in \mathbb{Z}, x, y \geq 0 \text{ and } bx + ay \leq abc\}$.

Problem 10.25 (102.-Met. Rec.)
Let α be irrational number. Prove that following function $f(x)$ is non periodic:
a) $f(x) = \sin \alpha x + \sin x$;
b) $f(x) = \sin \alpha x + \cos x$;
c) $f(x) = \tan \alpha x + \tan x$;
d) $f(x) = \tan \alpha x + \sin x$.

Problem 10.26 (103.-Met.Rec)

Let $a_1 = \dfrac{1}{2}, a_{n+1} = a_n + a_n^2$. Determine $\left\lfloor \dfrac{1}{a_1+1} + \dfrac{1}{a_2+1} + ... + \dfrac{1}{a_n+1} \right\rfloor$ for $n \geq 2$.

Problem 10.27 (Austria − Poland, 1980).
Given numerical sequence which for any $k, m \in \mathbb{N}$ satisfies to inequality $|a_{m+k} - a_k - a_m| \leq 1$.
Prove that for any $p, q \in \mathbb{N}$ holds inequality $\left|\dfrac{a_p}{p} - \dfrac{a_q}{q}\right| < \dfrac{1}{p} + \dfrac{1}{q}$.

Problem 10.28 (M.1195 ZK Proposed by ,Proposed by O.T.Izhboldin)

Prove that if sequence (a_n) satisfied to condition $|a_{n+m} - a_n - a_m| \leq \dfrac{1}{n+m}$,then (a_n) is arithmetic progression.

★Problem 10.29 (3571,CRUX,2010)
For given natural $n \geq 2$, among increasing arithmetic progression $x_1, x_2, ..., x_n$ such that
$x_1^2 + x_2^2 + ... + x_n^2 = 1$, find arithmetic progression with greates common difference d.

Problem 10.30.Quickies-Q2(CRUX?)
What is the units digit of the real number
$\left(15 + \sqrt{220}\right)^{2004} + \left(15 + \sqrt{220}\right)^{2005}$?

Part III

Solutions.

1. Divisibility.

Problem 1.1

Noting that $144 = 12^2$ and $1444 = 38^2$ we will prove that there are no other squares among numbers $a_n = 1\underbrace{44\ldots\ 4}_{n\ times}$.

Since $a_n = 10^n + \dfrac{4(10^n - 1)}{9} = \dfrac{13 \cdot 10^n - 4}{9}$ suffice to prove that $13 \cdot 10^n - 4$ can't be a perfect square for $n \geq 4$.Let $n \geq 4$ and assume that $13 \cdot 10^n - 4 = t^2 \iff 13 \cdot 10^n = t^2 + 4$ for some $t \in \mathbb{N}$.

Since $n \geq 4$ then $13 \cdot 10^n \vdots 16 \implies t^2 + 4 \vdots 16$.

Since $t^2 + 4 \vdots 16 \implies t^2 + 4 \vdots 4$ then $t = 2k$ for some k and , therefore, $t^2+4 \vdots 16 \iff 4\left(k^2+1\right) \vdots 16 \iff k^2+1 \vdots 4$ imply k is odd, that is $k = 2l+1$.

Then $t^2 + 4 = (4l+2)^2 + 4 = 16l^2 + 16l + 4$ isn't divisible by 16 and this contradict to $t^2 + 4 \vdots 16$.

Using modular notation we have $t^2 \equiv -4 \pmod{16} \implies t = 2k$ and, therefore, $t^2 \equiv -4 \pmod{16} \iff 4k^2 \equiv -4 \pmod{16} \iff k^2 \equiv -1 \pmod 4$ but that impossible because for $k \equiv 0, 1, 2, 3 \pmod 4$ we have $k^2 \equiv 0, 1 \pmod 4$.(This kind of solution we call "Reduction by modulo 16".)

Problem 1.2

This problem can be solved by reduction modulo 13. Indeed, since $385 \equiv -1 \pmod{13}$ and $18^2 \equiv 5^2 \pmod{13} \equiv 1 \pmod{13}$ then $385^{1980} \equiv 1 \pmod{13}$ and $18^{1980} = \left(18^2\right)^{940} \equiv 1 \pmod{13}$ .Hence, $385^{1980} + 18^{1980} \equiv 2 \pmod{13}$ .There is no natural t such that $t^2 \equiv 2 \pmod{13}$. Indeed, since $t \equiv r \pmod{13}$, where $r \in \{0, \pm1, \pm2, ..., \pm6\}$ then $t^2 \equiv r^2 \pmod{13}$ and $r^2 \pmod{13} \in \{0, \pm1, \pm3, \pm4\}$.

But $2 \pmod{13} \notin \{0, \pm1, \pm3, \pm4\}$.

Addition.

Note that natural numbers represented in form $3k+2, 5k \pm 2, 7k+3, 7k+5, 7k+6, 11k+r$,where $r \in \{2, 6, 7, 8, 10\}$ can't be a perfect square.

Problem 1.3

Let $f_n = f \circ f \circ ... \circ f$ (n-time composition). We should prove that $\gcd(f_n(a), f_m(a)) = 1$ for any $n, m \in \mathbb{N}$ and $n \neq m$.

First note that $\gcd(x, f(x)) = 1$.Indeed, $\gcd(x, f(x)) = \gcd\left(x, x^3 - x + 1\right) = \gcd(x, -x+1) = 1$.Suffice to prove that $\gcd(x, f_n(x)) = 1$ for any $n \in \mathbb{N}$ and any integer x. Note that $f_n(x)$ can be represented in the form $f_n(x) = xP_n(x) + 1$where $P_n(x)$ some polynomial with integer coefficients.

Indeed,

$$P_1(x) = x^2 - 1 \ and f_{n+1}(x) = xP_{n+1}(x) + 1 = f(f_n(x)) =$$

$$(xP_n(x)+1)^3 - (xP_n(x)+1) + 1 = x\left(x^2P_n^3(x) + 3xP_n^2(x) + 2P_n(x)\right) + 1 \implies$$

$$P_{n+1}(x) = x^2P_n^3(x) + 3xP_n^2(x) + 2P_n(x).$$

So, $\gcd(x, f_n(x)) = \gcd(x, xP_n(x) + 1) = \gcd(x, 1) = 1$.(Here was used again the **Preservation Lemma (**see solution to the **Problem 6.1))**

Problem 1.4

Consider 2 cases.

1. $x \leq 27$.Then

$$4^{27} + 4^{1000} + 4^x = y^2 \iff 2^{2x}\left(1 + 2^{54-2x} + 2^{2000-2x}\right) = y^2 \implies$$

$$y = 2^x a \implies 1 + 2^{54-2x} + 2^{2000-2x} = a^2.$$

Since $1+2^{54-2x}+2^{2000-2x} > 2^{2000-2x}$ and $1+2^{54-2x}+2^{2000-2x} < 1+2\cdot 2^{1000-x}+2^{2000-2x} \iff 2^{54-2x} < 2^{1001-x}$ then $2^{2000-2x} < a^2 < \left(1 + 2^{1000-x}\right)^2 \iff 2^{1000-x} < a < 1 + 2^{1000-x}$ that is the contradiction.

2. Let $27 < x$. Then,

$$4^{27} + 4^{1000} + 4^x = y^2 \iff 2^{54}\left(1 + 2^{2x-54} + 2^{1946}\right) = y^2 \implies$$

$$y = 2^{27} a \implies 1 + 2^{2x-54} + 2^{1946} = a^2.$$

Note that $1 + 2^{2x-54} + 2^{1946} = 1 + 2^{1946} + 2^{2x-54} = 1 + 2\cdot 2^{1945} + \left(2^{x-27}\right)^2$ be perfect square if $1945 = x - 27 \iff x = 1972$.For any $x > 1972$ we have $2^{2x-54} < a^2$ and $a^2 < 1+2^{x-26}+2^{2x-54} = 1+2\cdot 2^{x-27}+2^{2x-54} = \left(1 + 2^{x-27}\right)^2$ since $2^{x-26} > 2^{1946} \iff x - 26 > 1946 \iff x > 1972$.

Hence, $\left(2^{x-27}\right)^2 < a^2 < \left(1 + 2^{x-27}\right)^2 \iff 2^{x-27} < a < 1 + 2^{x-27}$, that is the contradicton.

So, there is no $x > 1972$ for which $4^{27} + 4^{1000} + 4^x$ is a perfect square and answer to the problem is $x = 1972$.

Problem 1.5

Suppose that there is m such that $5^n - 4^n = m^2$.Then $m \equiv 1 \pmod 2 \iff m = 2k + 1$ and, therefore, $5^n = 4^n + 4k(k+1) + 1 \implies 5^n \equiv 1 \pmod 8 \iff n = 2t, t \in \mathbb{N}$.

Thus, $5^n - 4^n = m^2 \iff 5^{2t} - 4^{2t} = m^2 \iff 5^{2t} - m^2 = 4^{2t} \iff$

$$\begin{cases} 5^t - m = 2^p \\ 5^t + m = 2^q \\ p + q = 4t, p < q \end{cases} \iff \begin{cases} 5^t = 2^{p-1} + 2^{q-1} \\ m = 2^{q-1} - 2^{p-1} \\ p + q = 4t, p < q \end{cases} \iff$$

$$\begin{cases} 5^t = 2^{p-1}\left(2^{q-p} + 1\right) \\ m = 2^{q-1} - 2^{p-1} \\ p + q = 4t, p < q \end{cases} \iff \begin{cases} 5^t = 2^{q-p} + 1 \\ m = 2^{q-1} - 2^{p-1} \\ p = 1, q = 4t - 1, t \in \mathbb{N} \end{cases} \iff \begin{cases} 5^t = 4^{2t-1} + 1 \\ m = 4^{2t-1} - 1 \end{cases}, t \in \mathbb{N}.$$

Since $n > 2$ then $t > 1$.But for any $t \geq 2$ holds inequality $5^t < 4^{2t-1}$(can be proved by MI). Thus, set $\{5^n - 4^n \mid n > 2\}$ is free from squares.

Problem 1.6

a) Suppose that there is m such that $2^n + 4^n = m^2$.

Then $2^n(1+2^n) = m^2 \implies \begin{cases} n = 2k \\ m = 2^k a \end{cases} \implies 1 + 2^{2k} = a^2 \implies$

$2^{2k} < a^2 < \left(2^k+1\right)^2 \iff 2^k < a < 2^k + 1$ that is the contradiction.

b*) First consider particular case.

Let $n = 0$. Then we obtain equation $1 + 4^m = k^2$ which have no solutions in $m \in \mathbb{N} \cup \{0\}$.

Indeed, since $k = 2p+1, p \in \mathbb{N}$ (because $p \geq 2$ is odd) then $1+4^m = k^2 \iff 4^m = p(p+1)$.

If p is odd it must be equal 1 (because prime decomposition of $p(p+1)$ is 2^{2m}).

Then $4^m = 2$,that is the contradiction;

If p is even then $p+1 > 1$ is odd and, therefore, has odd prime divisor – that is the contradiction again. Thus, $1 + 4^m$ can't be a perfect square.

Then for further we can assume that $n \in \mathbb{N}$. We will prove that if $2^n + 4^m = k^2$ for some $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ then $n > 2m$.

Indeed, assume that $n \leq 2m$ we obtain $2^n + 4^m = k^2 \iff 2^n\left(1 + 2^{2m-n}\right) = k^2$.

If $n = 2m$ then $2^{n+1} = k^2$ that is a contradiction because $n+1$ is odd;

If $n < 2m$ then $1+2^{2m-n}$ is odd and, therefore, from $2^n\left(1 + 2^{2m-n}\right) = k^2$ follows that $n = 2p$ for some natural p and $k = 2^p q$ for some odd q. Hence, $1 + 2^{2(m-p)} = q^2 \iff 1 + 4^{m-p} = q^2$.But $1 + 4^{m-p}$ can't be a perfect square.

Since $n > 2m$ then $2^n + 4^m = k^2 \iff 4^m\left(2^{n-2m}+1\right) = k^2$ and, therefore, $2^{n-2m}+1$ is a pefect square. Consider equation $2^p+1 = q^2 \iff 2^p = (q-1)(q+1)$ where $p \in \mathbb{N}$ and $q > 1$ is odd. Then $q-1 = 2^a, q+1 = 2^b$, where $b > a$ and $b + a = p$.

We have $q = 2^{b-1} + 2^{a-1}, 2^{b-1} - 2^{a-1} = 1 \iff 2^{a-1}\left(2^{b-a}-1\right) = 1 \iff$

$\begin{cases} a = 1 \\ b - a = 1 \end{cases} \iff \begin{cases} a = 1 \\ b = 2 \end{cases} \implies \begin{cases} q = 3 \\ p = 3 \end{cases}$.

Thus, $2^p+1 = q^2 \iff \begin{cases} q = 3 \\ p = 3 \end{cases}$ and, therefore, $2^n + 4^m$ is a perfect square iff $n = 2m+3$ and $m \in \mathbb{N} \cup \{0\}$.

Problem 1.7

We should find all n for which $3^n + 55 = m^2$ for some m.

Case 1. Let $n = 2k+1$ then $3^{2k+1} + 55 = m^2 \implies m^2 \equiv 2 \pmod 4$.

But that impossible because for any m holds $m \equiv 0, 1 \pmod 4$.

Case 2. If $n = 2k$ then $3^{2k} + 55 = m^2 \implies 3^{2k} < m^2$.

For k such that

$$3^{2k} + 55 < \left(3^k+1\right)^2 \iff 55 < 2 \cdot 3^k + 1 \iff 27 < 3^k \iff 4 \leq k$$

we have $3^{2k} < m^2 = 3^{2k} + 55 < \left(3^k+1\right)^2 \implies 3^k < m < 3^k + 1$,that is the contradiction.

Remains consider $k = 1, 2, 3$.
If $k = 1$ then $3^{2k} + 55 = 9 + 55 = 64 \implies n = 2, m = 8$
If $k = 2$ then $3^{2k} + 55 = 81 + 55 = 136$
If $k = 3$ then $3^{2k} + 55 = 3^6 + 55 = 784 = 28^2 \implies n = 6, m = 28$.
Answer: $n = 2, 6$.
(*Variant of the problem.*
Find all $n \in \mathbb{N}$ such that $a^n + b$ is perfect square if:
a) $a = 4, b = 5$;
b) $a = 8, b = 9$;
c) $a = 3, b = 55$).

Problem 1.8
a)
Solution 1.(Elementary with Math Induction)
Note that $a_0 = 3^{2^{4\cdot 0+1}} + 2 = 11, a_1 = 3^{2^{4\cdot 1+1}} + 2 = 3^{32} + 2 = 3^{32} - 9 + 11 = 9\left(3^{30} - 1\right) + 11$ and $3^{30} - 1 = \left(3^5 - 1\right)\left(3^{25} + 3^{20} + ... + 3^5 + 1\right)$ divisible by 11 because $3^5 - 1 = 243 - 1 = 11^2 \cdot 2$.

And we will prove, using Math Induction, that a_n divisible by 11 for any $n \in \mathbb{N}$.

Since Base of Math Induction already provided, remains the to prove (Step of MI) namely, for any $n \in\in \mathbb{N}$ in supposition that $11 \mid a_n$ we will prove $11 \mid a_{n+1}$.

We have $a_{n+1} = 3^{2^{4n+5}} + 2 = 3^{2^{4n+1}\cdot 16} + 2 = \left(\left(3^{2^{4n+1}} + 2\right) - 2\right)^{16} + 2 = (a_n - 2)^{16} + 2$. Since

$$(a_n - 2)^{16} = a_n^{16} - \binom{16}{1} a_n^{15} \cdot 2 + \binom{16}{2} a_n^{14} \cdot 2^2 - ... - \binom{16}{15} a_n \cdot 2^{15} + 2^{16}$$

and $11 \mid a_n$ remains to prove $11 \mid \left(2^{16} + 2\right) \iff 11 \mid \left(2^{15} + 1\right)$.

We have $2^{15} + 1 = \left(2^5 + 1\right)\left(2^{10} - 2^5 + 1\right) = 3 \cdot 11 \cdot \left(2^{10} - 2^5 + 1\right)$.

Solution 2.(Elementary, using factorization
$a^n - b^n = (a - b)\, a^{n-1} + a^{n-2}b + ... + b^n$**).**

Since $rem_{11}\left(3^5\right) = 1$ $(3^5 - 1 = 243 - 1 = 11^2 \cdot 2)$ we will find $rem_5\left(2^{4n+1}\right)$.

We have $2^{4n+1} = 2\left(16^n - 1\right) + 2$ and since $16^n - 1 = (16 - 1)\left(16^{n-1} + ... + 1\right)$ divisible by 5 then $2^{4n+1} = 5k + 2$ for some natural k.

Hence, $a_n = 3^{5k+2} + 2 = 3^{5k+2} - 9 + 11 = 9\left(3^{5k} - 1\right) + 11 = 9\left(3^{5k} - 1\right) + 11 = 9\left(3^5 - 1\right)\left(3^{5(k-1)} + ... + 1\right) + 11 = 18 \cdot 11^2\left(3^{5(k-1)} + ... + 1\right) + 11$.

Solution 3.(Academic)
Since by Little Fermat Theorem $3^{10} \equiv 1 \pmod{11}$ and $2^{4n} \equiv 1 \pmod{5}$ (because $2^4 \equiv 1 \pmod{5}$) yield $2^{4n+1} \equiv 2 \pmod{10}$ then $2^{4n+1} = 10k + 2$ for some $k \in \mathbb{N}$ and, therefore, $3^{2^{4n}} + 2 = 3^{10k+2} + 2 \equiv 3^2 + 2 \equiv 0 \pmod{11}$.

b) Since $2^{3^{4n+1}} + 3 = 2^{3^{4n+1}} - 8 + 11 = 8^{3^{4n}} - 8 + 11 = 8\left(8^{3^{4n}-1} - 1\right) + 11$ then $11 \mid b_n \iff 11 \mid 8^{3^{4n}-1} - 1$.

Also note that $3^{4n} - 1 = \left(3^4 - 1\right)\left(3^{4(n-1)} + 3^{4(n-2)} + ... + 1\right) = 80\left(3^{4(n-1)} + 3^{4(n-2)} + ... + 1\right) = 10k$,

where $k := 8\left(3^{4(n-1)} + 3^{4(n-2)} + ... + 1\right)$.
Hence, $8^{3^{4n}-1}-1 = 2^{30k}-1 = \left(2^{10}\right)^{3k}-1 = \left(2^{10}-1\right)\left(\left(2^{10}\right)^{3k-1} + \left(2^{10}\right)^{3k-2} ... + 1\right) =$ $1023\cdot\left(\left(2^{10}\right)^{3k-1} + \left(2^{10}\right)^{3k-2} ... + 1\right) = 11\cdot 93\left(\left(2^{10}\right)^{3k-1} + \left(2^{10}\right)^{3k-2} ... + 1\right)$.

c) Note that $c_0 = 13, c_1 = 2^{3^5}+5 = 2^{243}+5 = 2^{243}-8+13 = 8\left(2^{240}-1\right)+$ 13 and $2^{240}-1$ divisible by 13 because $2^6 = 65-1 = 5\cdot 13-1$ implies $13 \mid 2^{12}-1$ and, therefore, $2^{240}-1 = \left(2^{12}\right)^{20}-1 = \left(2^{12}-1\right)\left(\left(2^{12}\right)^{19} + \left(2^{12}\right)^{18} + ... + 1\right) =$ $13k$ for some $k \in \mathbb{N}$.

We will prove that $13 \mid c_n$ for any $n \in \mathbb{N}$.

Since $2^{12} \equiv 1 \pmod{13}$ and $3^{4n} \equiv 1 \pmod 4$ (because $3^2 \equiv 1 \pmod 4$) yield $3^{4n+1} \equiv 3 \pmod{12}$ then $3^{4n+1} = 12k+3$ for some $k \in \mathbb{N}$ and, therefore, $c_n = 2^{3^{4n+1}} + 5 = 2^{12k+3} + 5 \equiv 2^3 + 5 \equiv 0 \pmod{13}$.

Problem 1.9.

Suppose that $5^n - 1$ is divisible by $4^n - 1$ for some $n \in \mathbb{N}$.Since $4^n - 1 \vdots 3$ then $5^n - 1$ is divisible by 3 as well, that is $5^n \equiv 1 \pmod 3 \iff 2^n \equiv 1 \pmod 3 \iff$ $n \equiv 0 \pmod 2$. So, $n = 2k$ and, therefore, $5^{2k} - 1 \vdots 4^{2k} - 1 \iff 25^k - 1 \vdots 16^k - 1 \implies 25^k - 1 \vdots 15 \implies 25^k - 1 \vdots 5 \iff 1 \vdots 5,$ that is contradiction.

Problem 1.10

Since $a^n b^n = c^n d^n$ then WLOG we can assume that $n = 1$. Let $k := \gcd(d,b)$ and $q := \dfrac{d}{k}, p := \dfrac{b}{k}$ then $d = kq, b = kp, \gcd(p,q) = 1$ and since $ab = cd \iff \dfrac{a}{c} = \dfrac{d}{b} = \dfrac{q}{p}$ then $a = tq, c = tp$.

Therefore, $a^2+b^2+c^2+d^2 = t^2q^2+k^2p^2+t^2p^2+k^2q^2 = \left(p^2+q^2\right)\left(k^2+t^2\right)$.

Problem 1-11

We will find $rem_{10}\left(2^{4n+2}\right)$.Since $2^{4n+2} \equiv (-1)^{2n+1} \pmod 5 \iff 2^{4n+2} \equiv -1 \pmod 5 \iff 2^{4n+2} \equiv 4 \pmod 5$ and $2^{4n+2} \equiv 0 \pmod 2$ then $2^{4n+2} \equiv 4 \pmod{10}$ and, therefore, $5^{3^{4m}} - 2^{2^{4n+2}} \equiv \left(5^1 - 2^4\right) \pmod{11} \equiv 0 \pmod{11}$.

Problem 1.12

Suppose that for some natural n there is natural m such that

$$(n+1)^2 + (n+2)^2 + ... + (n+1000)^2 = m^2 \iff$$

$$1000n^2 + 2n(1+2+...+1000) + 1^2 + 2^2 + ... + 1000^2 = m^2 \iff$$

$$1000n^2 + n\cdot 1000\cdot 1001 + \frac{1000\cdot 1001\cdot 2001}{6} = m^2 \iff$$

$$1000n^2 + n \cdot 1000 \cdot 1001 + 500 \cdot 1001 \cdot 667 = m^2.$$

Since m^2 divisible by 500 then m divisible by 50, that is $m = 50k$ for some $k \in \mathbb{N}$ and, therefore,

$$1000n^2 + n \cdot 1000 \cdot 1001 + 500 \cdot 1001 \cdot 667 = 2500k^2 \iff$$

$$2n^2 + 2002n + 1001 \cdot 667 = 5k^2 \implies 2n^2 + 2n + 2 \equiv 0 \pmod 5 \iff$$

$4n^2+4n+4 \equiv 0 \pmod 5 \iff (2n+1)^2 \equiv 2 \pmod 5$ that is contradiction, because there are no squares of integers which is congruent 2 by modulo 5. Indeed, for $r \pmod 5 \in \{0, \pm 1, \pm 2\}$ we have $r^2 \pmod 5 \in \{0, \pm 1\}$.

Problem 1.13

Let $2n+1 = q^2, 3n+1 = p^2$.Since q^2 odd then $q = 2k+1$ for some $k \in \mathbb{Z}$ and, therefore, $2n+1 = (2k+1)^2 \iff n = 2k(k+1)$.Hence, $p^2 = 3 \cdot 2k(k+1) + 1 = 6k^2 + 6k + 1 \implies p$ is odd, that is $p = 2t+1$, for some integer t.Therefore, $p^2 = 6k^2 + 6k + 1 \iff (2t+1)^2 = 6k^2 + 6k + 1 \iff$

$2t(t+1) = 3k(k+1)$. Then $3n = 6k(k+1) = 4t(t+1) \,\vdots\, 8 \implies n \,\vdots\, 8$.

Since $p^2+q^2 = 5n+2$ then $p^2+q^2 \equiv 2 \pmod 5 \iff \begin{cases} p^2 \equiv 1 \pmod 5 \\ q^2 \equiv 1 \pmod 5 \end{cases} \implies$

$n = p^2 - q^2 \equiv 0 \pmod 5$.Thus, $n \,\vdots\, 40$.

Problem 1.14

Let $S(n)$ be sum of digits of natural number n. Assume that there is $a \in \mathbb{N}$ such that $S(a^2) = 1985$. Then $a^2 \equiv S(a^2) \pmod 3 \equiv 1985 \pmod 3 \equiv -1 \pmod 3$ but that isn't possible because for any integer a we have $a \equiv 0, \pm 1 \pmod 3 \implies a^2 \equiv 0, 1 \pmod 3$.

Problem 1.15.

Let $2^n + 4^m = t^2$.Firstly we will prove that $n \neq 2m$. Suppose that $n = 2m$.Then $2^{2m} + 4^m = t^2 \iff 2^{2m+1} = t^2$,that is the contradiction.

Consider case $n < 2m$.Then $2^n + 4^m = t^2 \iff 2^n\left(1 + 2^{2m-n}\right) = t^2 \implies$

$\begin{cases} n = 2p \\ t = 2^p q \end{cases}$ for some natural p and odd q.

Hence, $1 + 2^{2(m-p)} = q^2 \implies 2^{2(m-p)} < q^2 < (1 + 2^{m-p})^2 \iff 2^{m-p} < q < 1 + 2^{m-p}$, that is the contradiction.

Let now $n > 2m$.Then $2^n + 4^m = t^2 \iff 2^{2m}\left(1 + 2^{n-2m}\right) = t^2$.Since n can't be even (because otherwise we get contradiction) then $n = 2p + 1$ for some natural $p \geq m$ and, therefore, $2^{2m}\left(1 + 2^{2(p-m)+1}\right) = t^2 \implies$

$\begin{cases} 1 + 2^{2(p-m)+1} = q^2 \\ t = 2^m q \end{cases}$ for some natural q.

Let $l := p-m$ then $l \geq 0$ and $1+2^{2l+1} = q^2 \iff 2^{2l+1} = (q-1)(q+1) \iff$

$$\begin{cases} q-1=2^a \\ q+1=2^b \\ 0\le a<b, a+b=2l+1 \end{cases} \implies \begin{cases} q=2^{b-1}+2^{a-1} \\ 1=2^{b-1}-2^{a-1} \\ 0\le a<b, a+b=2l+1 \end{cases} \iff \begin{cases} q=3 \\ a=1, b=2, l=1 \end{cases}$$

.

Hence, $m=p-1,\ n=2p+1$ and $t=3\cdot 2^{p-1}$,for any natural p,that is $2^{2p+1}+4^{p-1}=\left(3\cdot 2^{p-1}\right)^2$.

Problem 1.16

a) Since $ord_p(n!)=\left[\dfrac{n}{2}\right]+\left[\dfrac{n}{2^2}\right]+...+\left[\dfrac{n}{2^k}\right]$ where k is such that $n\ge 2^k$ and $n<2^k$ then

$$ord_p(n!)\le \frac{n}{2}+\frac{n}{2^2}+...+\frac{n}{2^k}=\frac{n}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{k-1}}\right)=$$

$$\frac{n}{2}\cdot\frac{1-1/2^k}{1-1/2}<\frac{n}{2}\cdot\frac{1}{1-1/2}=n$$

.

b) Since $\dfrac{(p-1)\,n}{p^k}<\dfrac{n}{p^{k-1}}, k=1,2,...$ then $\left[\dfrac{(p-1)\,n}{p^k}\right]\le\left[\dfrac{n}{p^{k-1}}\right]$ and, therefore,

$$ord_p(((p-1)\,n)!)=\sum_{k=1}^{k_{\max}}\left[\frac{(p-1)\,n}{p^k}\right]\le n+\sum_{k=2}^{k_{\max}}\left[\frac{n}{p^{k-1}}\right]=n+ord_p(n!)$$

c) By Legendre formula $ord_p((n!)!)=\sum\limits_{k=1}^{k_{\max}}\left[\dfrac{n!}{2^k}\right]$, where $2^{k_{\max}}\le n!$ and $2^{k_{\max}+1}>n!$

Since $\dfrac{n!}{2^k}=\dfrac{(n-1)!n}{2^k}\ge n\left[\dfrac{(n-1)!}{2^k}\right]$ then $\left[\dfrac{n!}{2^k}\right]\ge n\left[\dfrac{(n-1)!}{2^k}\right]$ and, therefore,

$$ord_p((n!)!)\ge n\sum_{k=1}^{k_{\max}}\left[\frac{(n-1)!}{2^k}\right]=n\cdot ord_p(((n-1)!)!)=ord_p((((n-1)!)!)^n).$$

d) Since $ord_p((pn)!)=\sum\limits_{k=1}^{k_{\max}}\left[\dfrac{n}{p^k}\right]=n+\sum\limits_{k=2}^{k_{\max}}\left[\dfrac{n}{p^{k-1}}\right]=n+ord_p(n!)$

then $ord_p\left(\dfrac{(pn)!}{n!}\right)=ord_p((pn)!)-ord_p(n!)=n.$

e) $(n!)!\ge((n-1)!)^{n!}\iff (n!)!\ge\left(\dfrac{n!}{n}\right)^{n!}\iff (n!)!n^{n!}\ge(n!)^{n!}$.

We will prove more general inequality $m!n^m\ge m^m$ for any $m,n\in\mathbb{N}, n\ge 3$.

Math Induction by $m\in\mathbb{N}, m\ge 3$.

Let $m=1$ then $1!n^1\ge 1^1\iff n\ge 1$;

$m=2$ then $2!n^{2!}\geq 2^2 \iff 2n^2\geq 4 \iff n^2\geq 2;$
$m=3$ then $3!n^{3!}\geq 3^3 \iff 6n^6\geq 27 \iff 2n^6\geq 9$ holds for $n\geq 3$.

Instead step of MI we will use multiplicative reduction, that is we will prove inequality

$$\textbf{(1)}\quad \frac{(m+1)!n^{m+1}}{m!n^m}\geq \frac{(m+1)^{m+1}}{m^m}, m\in\mathbb{N}.$$

We have

$$\textbf{(1)} \iff (m+1)\,n\geq \frac{(m+1)^{m+1}}{m^m} \iff n\geq \frac{(m+1)^m}{m^m} \iff n\geq\left(1+\frac{1}{m}\right)^m \Longleftarrow n>e>\left(1+\frac{1}{m}\right)^m.$$

Applying inequality $m!n^m\geq m^m$ for $m=n!, n\geq 3$ we obtain inequality $(n!)!\geq ((n-1)!)^{n!}$ for any $n\geq 3$.

If $n=1$ then $(1!)!\geq ((1-1)!)^{1!} \iff 1=1$.
If $n=2$ then $(2!)!\geq ((2-1)!)^{2!} \iff 2\geq 1^2=1$ also holds.
So, $(n!)!\geq ((n-1)!)^{n!}$ holds for any $n\in\mathbb{N}$.

Problem 1.17

a) Suffice to prove that $ord_p\left((n!)!\right)\geq ord_p\left((n!)^{(n-1)!}\right)=(n-1)!{\cdot}ord_p\left(n!\right)$ for any prime p.

By Legendre formula

$$ord_p\left((n!)!\right)=\sum_{k=1}^{k_{\max}}\left[\frac{n!}{p^k}\right],\ where p^{k_{\max}}\leq n!\ and p^{k_{\max}+1}>n!.$$

Since $\dfrac{n!}{p^k}=\dfrac{(n-1)!n}{p^k}\geq (n-1)!\left[\dfrac{n}{p^k}\right]$ then $\left[\dfrac{n!}{p^k}\right]\geq (n-1)!\left[\dfrac{n}{p^k}\right]$ and, therefore,

$$ord_p\left((n!)!\right)\geq (n-1)!\sum_{k=1}^{k_{\max}}\left[\frac{n}{p^k}\right]=(n-1)!\cdot ord_p\left(n!\right)=(n-1)!\cdot ord_p\left(n!\right).$$

b) Since $\dfrac{n!}{p^k}=\dfrac{(n-1)!n}{p^k}\geq n\left[\dfrac{(n-1)!}{p^k}\right]$ then

$$\left[\frac{n!}{p^k}\right]\geq n\left[\frac{(n-1)!}{p^k}\right] \implies$$

$$ord_p\left((n!)!\right)\geq n\sum_{k=1}^{k_{\max}}\left[\frac{(n-1)!}{p^k}\right]=n\cdot ord_p\left((n-1)!\right)=ord_p\left(\ ((n-1)!)!^n\right) \implies$$

$$(n!)!\ \vdots\ ((n-1)!)!^n.$$

c) $\dfrac{n^n}{p^k} = n^{n-1} \cdot \dfrac{n}{p^k} \geq n^{n-1}\left[\dfrac{n}{p^k}\right] \implies \left[\dfrac{n^n}{p^k}\right] \geq n^{n-1}\left[\dfrac{n}{p^k}\right]$ and, therefore,

$$ord_p\left((n^n)!\right) \geq n^{n-1} ord_p\left(n!\right).$$

e) $\dfrac{n^{m+k}}{p^i} = n^k \cdot \dfrac{n^m}{p^i} \geq n^k\left[\dfrac{n^m}{p^i}\right] \implies \left[\dfrac{n^{m+k}}{p^i}\right] \geq n^k\left[\dfrac{n^m}{p^k}\right]$ and,therefore,

$$ord_p\left(\left(n^{m+k}\right)!\right) \geq n^k \cdot ord\left(n^m!\right)$$

f) $\dfrac{n \cdot m}{p^i} \geq m\left[\dfrac{n}{p^i}\right] \implies \left[\dfrac{n \cdot m}{p^i}\right] \geq m\left[\dfrac{n}{p^i}\right]$ and,therefore,

$$ord_p\left((n \cdot m)!\right) \geq m \cdot ord_p\left(n!\right).$$

g) First we will prove that $ord_2\left((2n)!\right) \geq ord_2\left(n!\right) + ord_2\left((n+1)!\right).$
To prove that suffice to prove inequality

$$\left[\frac{n}{2^{k-1}}\right] \geq \left[\frac{n}{2^k}\right] + \left[\frac{n+1}{2^k}\right], k = 1, 2, ..., .$$

For $k = 1$ holds equality $n = \left[\dfrac{n}{2}\right] + \left[\dfrac{n+1}{2}\right].$
Let $k > 1$.Then $\dfrac{n}{2^{k-1}} = \dfrac{1}{2^{k-1}}\left[\dfrac{n}{2}\right] + \dfrac{1}{2^{k-1}}\left[\dfrac{n+1}{2}\right].$
Since for any positive x and natural m we have

$$\frac{x}{m} \geq \left[\frac{x}{m}\right] \iff x \geq m\left[\frac{x}{m}\right] \implies [x] \geq m\left[\frac{x}{m}\right] \iff \frac{[x]}{m} \geq \left[\frac{x}{m}\right]$$

then

$$\frac{1}{2^{k-1}}\left[\frac{n}{2}\right] \geq \left[\frac{n}{2^k}\right] \; and \frac{1}{2^{k-1}}\left[\frac{n+1}{2}\right] \geq \left[\frac{n+1}{2^k}\right].$$

Hence,

$$\frac{n}{2^{k-1}} = \frac{1}{2^{k-1}}\left[\frac{n}{2}\right] + \frac{1}{2^{k-1}}\left[\frac{n+1}{2}\right] \geq \left[\frac{n}{2^k}\right] + \left[\frac{n+1}{2^k}\right] \implies$$

$$\left[\frac{n}{2^{k-1}}\right] \geq \left[\frac{n}{2^k}\right] + \left[\frac{n+1}{2^k}\right]$$

and, therefore, $ord_2\left((2n)!\right) \geq ord_2\left(n!\right) + ord_2\left((n+1)!\right).$

Let now $p > 2$.Note that $2n \geq p\left(\left[\dfrac{n}{p}\right] + \left[\dfrac{n+1}{p}\right]\right)$.Indeed, let $n = lp + r$,where $0 \leq r \leq n-1$.If $r < n-1$ then $\left[\dfrac{n}{p}\right] = \left[\dfrac{n+1}{p}\right] = k$ and, therefore, $p\left(\left[\dfrac{n}{p}\right] + \left[\dfrac{n+1}{p}\right]\right) = 2pk \leq 2pk + 2r = 2n$.If $r = p-1$ then $\left[\dfrac{n}{p}\right] =$

k, $\left[\dfrac{n+1}{p}\right]=k+1$ and, therefore,

$$p\left(\left[\frac{n}{p}\right]+\left[\frac{n+1}{p}\right]\right)=2pk+p<2pk+2(p-1)$$

because $2(p-1)>p \iff p>2$.
Since

$$2n\geq p\left(\left[\frac{n}{p}\right]+\left[\frac{n+1}{p}\right]\right) \implies \frac{2n}{p^k}\geq\frac{1}{p^{k-1}}\left[\frac{n}{p}\right]+\frac{1}{p^{k-1}}\left[\frac{n+1}{p}\right]$$

then applying inequality $\dfrac{[x]}{m}\geq\left[\dfrac{x}{m}\right]$ we obtain

$$\frac{2n}{p^k}\geq\frac{1}{p^{k-1}}\left[\frac{n}{p}\right]+\frac{1}{p^{k-1}}\left[\frac{n+1}{p}\right]\geq\frac{2n}{p}\geq\left[\frac{n}{p^k}\right]+\left[\frac{n+1}{p^k}\right] \implies$$

$$\left[\frac{2n}{p}\right]\geq\left[\frac{n}{p^k}\right]+\left[\frac{n+1}{p^k}\right]$$

and, and, therefore,
$ord_p\left((2n)!\right)\geq ord_p\left(n!\right)+ord_p\left((n+1)!\right)$.

Combinatorial solution.
Note that $\dfrac{(2n)!}{n!\,(n+1)!}=\dfrac{\binom{2n}{n}}{n+1}$. Since $\gcd(n,n+1)=1$ then $\binom{2n}{n}\,\vdots\,(n+1)$ iff $n\binom{2n}{n}\,\vdots\,(n+1) \iff \dfrac{n\binom{2n}{n}}{n+1}$ is integer. But, $\dfrac{n\binom{2n}{n}}{n+1}=\binom{2n}{n-1}\in\mathbb{N}$.

h) $(n+1)(n+2)\ldots(n+m)\,\vdots\,m!$ for any n,m.
First solution is combinatorial:

$$\frac{(n+1)(n+2)\ldots(n+m)}{m!}=\binom{n+m+1}{m}.$$

Second solution.
Since $\dfrac{(n+1)(n+2)\ldots(n+m)}{m!}=\dfrac{(n+m)!}{n!m!}$ we will prove that

$$ord_p\left(\frac{(n+m)!}{n!m!}\right)\geq 0 \iff ord_p\left((n+m)!\right)\geq ord_p\left(n!\right)+ord_p\left(m!\right).$$

Since $[x+y]\geq[x]+[y]$ then $\left[\dfrac{n+m}{p^k}\right]\geq\left[\dfrac{n}{p^k}\right]+\left[\dfrac{m}{p^k}\right]$ and, therefore,

$$ord_p\left((n+m)!\right)\geq ord_p\left(n!\right)+ord_p\left(m!\right).$$

Problem 1.18

Since $S_n = \dfrac{n(n+1)}{2}$ then $S_n \equiv 1 \pmod 5 \iff n(n+1) \equiv 2 \pmod 5 \iff$

$$n^2 + n - 2 \equiv 0 \pmod 5 \iff (n+2)(n-1) \equiv 0 \pmod 5 \iff$$

$$\left[\begin{array}{l} n+2 \equiv 0 \pmod 5 \\ n-1 \equiv 0 \pmod 5 \end{array}\right. \iff \left[\begin{array}{l} n \equiv -2 \pmod 5 \\ n \equiv -4 \pmod 5 \end{array}\right. .$$

Thus, $n = 5k-2, k \in \mathbb{N}$ or $n = 5k-4, k \in \mathbb{N}$.
Another variant of solution:
Since 5 is prime then

$$\frac{S_n - 1}{5} \in \mathbb{Z} \iff \frac{n(n+1)-2}{2 \cdot 5} \in \mathbb{Z} \implies \frac{n(n+1)-2}{5} \in \mathbb{Z} \iff (n+2)(n-1) \vdots 5 \iff$$

$$\left[\begin{array}{l} n+2 \vdots 5 \\ n-1 \vdots 5 \end{array}\right. \iff \left[\begin{array}{l} n+2 \vdots 5 \\ n+4 \vdots 5 \end{array}\right. \iff \left[\begin{array}{l} n = 5k-2, k \in \mathbb{N} \\ n = 5k-4, k \in \mathbb{N} \end{array}\right. .$$

Problem 1.19

Let $a_n := \left(\sqrt{3}+1\right)^{2n} + \left(\sqrt{3}-1\right)^{2n} = \left(4+2\sqrt{3}\right)^n + \left(4-2\sqrt{3}\right)^n$. Then a_n satisfy to the recurrence $a_{n+1} - 8a_n + 4a_{n-1} = 0$ and $a_0 = 2, a_1 = 8$.So, a_n is integer for all $n \in \mathbb{N}$. Since $\left(4-2\sqrt{3}\right)^n \in (0,1)$ then from representation $\left(\sqrt{3}+1\right)^{2n} = a_n - \left(\sqrt{3}-1\right)^{2n}$ and $\left(\sqrt{3}+1\right)^{2n} \in (a_n - 1, a_n)$ follows that $a_n = \left\lceil \left(\sqrt{3}+1\right)^{2n} \right\rceil$ ($\lceil x \rceil$ is ceiling of x)

We see that $a_n \vdots 2^{n+1}$ for $n = 0, 1$.From supposition $a_n \vdots 2^{n+1}, a_{n-1} \vdots 2^n$ and $a_{n+1} = 8a_n - 4a_{n-1} = 4(2a_n - a_{n-1})$ immediately follows that $a_{n+1} \vdots 2 \cdot 2^n$.

Let now $b_n := \dfrac{a_n}{2^{n+1}}$ then $b_0 = 1, b_1 = 2$ and $b_{n+1} = 4b_n - b_{n-1}$.
Since $b_{n+1} \equiv b_{n-1} \pmod 2$ then $b_{2m} \equiv 1 \pmod 2$ and $ord_2(a_{2m}) = 2m+1$.
($a_{2m} = 2^{2m+1} b_{2m}$,where b_{2m} odd for any $m \in \mathbb{N} \cup \{0\}$.)

2. Diophantine equation.

Problem 2.1

We will solve equation

$$y = \frac{3x - \sqrt{9x^2 + 160x + 800}}{16} \iff 9x^2 + 160x + 800 = (3x - 16y)^2 \iff$$

$$5x + 3xy - 8y^2 + 25 = 0 \iff x = \frac{8y^2 - 25}{3y + 5}.$$

Note that

$$\gcd\left(8y^2-25,3y+5\right)=\gcd\left(y^2+15y+25,3y+5\right)=\gcd\left(3y^2+45y+75,3y+5\right)=$$

$$\gcd\left(40y+75,3y+5\right)=\gcd\left(y+10,3y+5\right)=\gcd\left(y+10,25\right)\in\{1,5,25\}.$$

Since x is integer iff $\gcd(y+10,25)=|3y+5|$ then possible three options: $|3y+5|=1 \iff y=-2$, $|3y+5|=5 \iff y=0$ and $|3y+5|=25 \iff y=-10$.

Hence for $y=-2$ we obtain $x=\dfrac{8\cdot 4-25}{-6+5}=-7$,for $y=0$ we obtain $x=-5$ and for $y=-6$ we obtain $x=\dfrac{8\cdot 100-25}{3\cdot(-10)+5}=-31$.

So, $\dfrac{3x-\sqrt{9x^2+160x+800}}{16}$ is integer only for $x=-5,-7,-31$.

Problem 2.2

Since $x\equiv x-2y \pmod 2$ then equation $x^2-2xy=1978$ have no sulutions in integers (x,y) with odd x because then $x^2-2xy=x(x-2y)$ is odd, and have no integer sulutions (x,y) with even x because then x^2-2xy is divisible by 4 but 1978 isn't divisible by 4.

Problem 2.3

Note that $2m^2+m=3n^2+n$ can be rewritten as

$$2m^2-2n^2+m-n=n^2 \iff (m-n)(2m+2n+1)=n^2$$

and as

$$3m^2-3n^2+m-n=m^2 \iff (m-n)(3m+3n+1)=m^2.$$

Note that if $m-n$ is a perfect square then it immediatelly imply that $2m+2n+1, 3m+3n+1$ are perfect squares as well. Thus, suffices to prove that $m-n$ is a perfect square for any natural n,m that satisfies to equation $2m^2+m=3n^2+n$.

Since $2m^2+m=3n^2+n \iff m(2m+1)=n(3n+1) \iff \dfrac{m}{n}=\dfrac{3n+1}{2m+1}$.

Let $\dfrac{a}{b}$ be irredusible fraction such that $\dfrac{3n+1}{2m+1}=\dfrac{m}{n}=\dfrac{a}{b}$.

Then $\dfrac{m}{n}=\dfrac{a}{b}$ and $\dfrac{3n+1}{2m+1}=\dfrac{a}{b} \iff \begin{cases} bm-an=0 \\ 3bn-2am=a-b \end{cases} \iff$
$\begin{cases} n=\dfrac{ab-b^2}{3b^2-2a^2} \\ m=\dfrac{a^2-ab}{3b^2-2a^2} \end{cases}.$

Since $d\left(3b^2-2a^2,a^2-ab\right)=d\left(3b^2-2ab,a^2-ab\right)=d(3b-2a,a-b)=$
$d(b,a-b)=d(b,a)=1$ and similarly $d\left(3b^2-2a^2,a^2-ab\right)=1$ then $n,m\in\mathbb{Z}$ iff $\left|3b^2-2a^2\right|=1$.

But $3b^2 - 2a^2 = -1$ have no solutions because

$$3b^2 - 2a^2 = -1 \implies a^2 \equiv -1 \pmod 3$$

and that isn't possible.

So, remainse $3b^2 - 2a^2 = 1$ which have infinitely many integer solutions.

Since all integer solutions of equation $2m^2 + m = 3n^2 + n$ can be represented in the form

$(m, n) = (a^2 - ab, ab - b^2)$ where a, b be any coprime numbers satisfying $3b^2 - 2a^2 = 1$

then $m - n = a^2 - ab - (ab - b^2) = (a - b)^2$.

Problem 2.4

Let $(x, y, z) \neq (0, 0, 0)$ is integer solution of equation $x^3 - 2y^3 - 4z^3 = 0$. Since x divisible by 2 then $x = 2x_1$ then

$$8x_1^3 - 2y^3 - 4z^3 = 0 \iff 4x_1^3 - y^3 - 2z^3 = 0 \implies y \vdots 2 \iff y = 2y_1 \implies$$

$$4x_1^3 - 8y_1^3 - 2z^3 = 0 \iff 2x_1^3 - 4y_1^3 - z^3 = 0 \implies z = 2z_1 \implies$$

$$2x_1^3 - 4y_1^3 - 8z_1^3 = 0 \iff x_1^3 - 2y_1^3 - 4z_1^3 = 0.$$

Thus, starting with non-zero solution (x, y, z) we obtain new non-zero solution $(x_1, y_1, z_1) = \dfrac{1}{2}(x, y, z)$. Similarly from (x_1, y_1, z_1) we obtain integer non-zero solution $(x_2, y_2, z_2) = \dfrac{1}{2}(x_1, y_1, z_1)$ and so on.....from $(x_n, y_n, z_n) \neq (0, 0, 0)$ we obtain $(x_{n+1}, y_{n+1}, z_{n+}1) = \dfrac{1}{2}(x_n, y_n, z_n)$, $n \in \mathbb{N}$. Hence, $(x_n, y_n, z_n) = \dfrac{1}{2^n}(x, y, z)$ be triple of integer numbers for any natural n.Since $|x| + |y| + |z| \in \mathbb{N}$ and $(x_n, y_n, z_n) \neq 0$ for any n then $|x_n| + |y_n| + |z_n| = \dfrac{|x| + |y| + |z|}{2^n}$ is infinite strictly decreasing sequence of natural numbers.

And that is the contradiction to Well Ordering Principle: Any non empty subset of natural numbers has the smallest element. So, equation $x^3 - 2y^3 - 4z^3 = 0$ have no non-trivial ineger solution.

Problem 2.5

We will find a solution in the form $(x, y, z) = (2^n, 2^m, 2^k)$ .By sbstitution in equation we obtain $2^{3n+1} + 2^{5m} = 2^{7k}$ and we claim

$$\begin{cases} 3n + 1 = 5m \\ 5m + 1 = 7k \end{cases} \iff \begin{cases} 3n + 2 = 7k \\ 5m + 1 = 7k \end{cases} \iff \begin{cases} 7k \equiv 2 \pmod 3 \\ 7k \equiv 1 \pmod 5 \end{cases}.$$

We have

$$\begin{cases} 7k \equiv 2 \pmod 3 \\ 7k \equiv 1 \pmod 5 \end{cases} \iff \begin{cases} k \equiv 2 \pmod 3 \\ 2k \equiv 1 \pmod 5 \end{cases} \iff \begin{cases} k \equiv 2 \pmod 3 \\ 2k \equiv -4 \pmod 5 \end{cases} \iff$$

$$\begin{cases} k \equiv 2 \pmod 3 \\ k \equiv -2 \pmod 5 \end{cases} \iff \begin{cases} k \equiv 2 \pmod 3 \\ k \equiv 3 \pmod 5 \end{cases} \iff \begin{cases} 5k \equiv 10 \pmod{15} \\ 3k \equiv 9 \pmod{15} \end{cases} \iff 2k \equiv 1 \pmod{15} \iff$$

$k \equiv 8 \pmod{15} \iff k = 15t + 8, t \in \mathbb{Z}$.

Then, $3n + 2 = 7k + 8 \iff 3n = 105t + 54 \iff n = 35t + 18$ and $5m + 1 = 7k + 8 \iff 5m = 105t + 55 \iff m = 21t + 11, t \in \mathbb{Z}$.

Thus, $(x, y, z) = \left(2^{35t+18}, 2^{21t+11}, 2^{15t+8}\right)$ is solution of equation $2x^3 + y^5 = z^7$ for any natural t.

Indeed, $2 \cdot \left(2^{35t+18}\right)^3 + \left(2^{21t+11}\right)^5 = 2^{105t+55} + 2^{105t+55} = 2^{105t+56} = \left(2^{15t+8}\right)^7$.

Problem 2.6 (44.Met.Rec)

We will find a solution in the form $(x, y, z, t, u, v) = (t, t, t, t, 2v, v)$ .By sbstitution in equation we obtain $4t^3 = 16v^4 - v^4 \iff 4t^3 = 15v^4$. If we can prove that equation $4t^3 = 15v^4$ has infinitely many natural solutions then original equation has infinitely many natural solutions as well.

We will find a solution of equation $4t^3 = 15v^4$ in the form $(t, v) = \left(4^a 15^b, 4^c 15^d\right), a, b, c, d \in \mathbb{N}$.

Then $4t^3 = 15v^4$ becomes $4^{3a+1} 15^{3b} = 4^{4c} 15^{4d+1} \iff \begin{cases} 3a + 1 = 4c \\ 3b = 4d + 1 \end{cases}$.

We have

$$3a + 1 = 4c \iff 4c - 3a = 1 \iff 4(c - 1) = 3(a - 1) \iff$$

$$\begin{cases} c - 1 = 3p \\ a - 1 = 4p \end{cases}, p \in \mathbb{Z} \iff \begin{cases} c = 3p + 1 \\ a = 4p + 1 \end{cases}, p \in \mathbb{Z}.$$

In particular, $\begin{cases} c = 3p - 2 \\ a = 4p - 3 \end{cases}, p \in \mathbb{N}$ give us infinitely many natural a, c.

Similarly, $3b = 4d + 1 \iff 3(b + 1) = 4(d + 1) \iff \begin{cases} b = 4q - 1 \\ d = 3q - 1 \end{cases}, q \in \mathbb{Z}$ and, in particular, $\begin{cases} b = 4q - 1 \\ d = 3q - 1 \end{cases}, q \in \mathbb{N}$ give us infinitely many natural b, d.

Thus, $(t, v) = \left(4^{4p-3} 15^{4q-1}, 4^{3p-2} 15^{3q-1}\right) .p, q \in \mathbb{N}$ give us infinitely many natural solutions (t, v) of equation $4t^3 = 15v^4$.Indeed, $4 \cdot \left(4^{4p-3} 15^{4q-1}\right)^3 = 4^{12p-8} 15^{12q-3}$ and $15 \cdot \left(4^{3p-2} 15^{3q-1}\right)^4 = 4^{12p-8} 15^{12q-3}$.

Problem 2.7

From experience of solutions to problems 2.5,2.6 where we used that all exponents are totaly coprime, can be impression that this equation have no natural solutions becuase all exponents aren't totaly coprime. But it isn't so. Here we can use another idea of solution.

We will find natural solution represented the form $(x, y, z, t) = \left(az^3, bz^2, z, cu\right)$.

For such (x, y, z, t) equation becomes $\left(a^4 + b^6 + 1\right) z^{12} = c^4u^4$ and we claim $a^4 + b^6 + 1$ be 4-th degree of some natural number. Easy to see that $a = 2, b = 2$ satisfy this requirement. Then for $(x, y, z, t) = \left(2z^3, 2z^2, z, cu\right)$ we have $81z^{12} = c^4u^4 \iff 3z^4 = cu$.Thus, we obtain infinitely many natural solutions of equation $x^4 + y^6 + z^{12} = t^4$ represented by quads $(x, y, z, t) = \left(2z^3, 2z^2, z, 3z^4\right), z \in \mathbb{N}$.

Problem 2.8

a),b),c)

Since for any integer number a can be represented in the form $a = 3k + r$ where $r \in \{0, 1, -1\}$ then $a^3 = 27k^3 + 27k^2r + 9kr^2 + r^3$ and, therefore, $a^3 \equiv r^3 \pmod 9$ for any $a \in \mathbb{Z}$. Thus, since $r \in \{0, 1, -1\}$ then $r^3 \in \{0, 1, -1\}$ as well. Hence, for $x, y, z \in \mathbb{Z}$ we have $x = 3k + p, y = 3l + q, z = 3m + r,$ where $p, q, r \in \{0, 1, -1\}$ and, therefore, $x^3 + y^3 \equiv p^3 + q^3 \pmod 9$ and $x^3 + y^3 + z^3 \equiv p^3 + q^3 + r^3 \pmod 9$ where $-2 \le p^3 + q^3 \le 2$ and $-3 \le p^3 + q^3 + r^3 \le 3$.Thus, $x^3 + y^3 \equiv \pm 4 \pmod 9$, $x^3 + y^3 \equiv \pm 3 \pmod 9$ and $x^3 + y^3 + z^3 \equiv \pm 4 \pmod 9$ impossible.

d) Since $117 \vdots 9$ then $x^3 + 117y^3 = 5 \implies x^3 \equiv 5 \pmod 9 \implies x^3 \equiv 2 \pmod 3 \implies x^3 \equiv 8 \pmod 9$,

but 5 isn't congruence to 8 by modulo 9.

Remark.(Just in case).

Since $x^3 + y^3 + z^3 = 3xyz + (x + y + z)^3 - 3(x + y + z)(xy + yz + zx)$ then $x^3 + y^3 + z^3 \equiv x + y + z \pmod 3$.)

Problem 2.9

Since

$$3a = x^2 + 2y^2 \implies x^2 + 2y^2 \equiv 0 \pmod 3 \iff x^2 \equiv y^2 \pmod 3 \iff |x| \equiv |y| \pmod 3 \iff$$

then

$$\left\{\begin{array}{c} x \equiv 0 \pmod 3, y \equiv 0 \pmod 3 \\ x \equiv \delta \pmod 3, y \equiv \sigma \pmod 3 \text{ where } |\sigma| = |\delta| = 1 \end{array}\right.$$

Case1. Let $x \equiv 0 \pmod 3, y \equiv 0 \pmod 3$. Then $x = 3p, y = 3q, p, q \in \mathbb{Z}$ and, therefore, $3a = 9p^2 + 18q^2 \iff a = 3\left(p^2 + 2q^2\right)$.

Note that $\left(n^2 + 2m^2\right)\left(p^2 + 2q^2\right) = n^2p^2 + 4m^2q^2 + 2n^2q^2 + 2m^2p^2 = n^2p^2 + 4nmpq + 4m^2q^2 + 2n^2q^2 - 4nmpq + 2m^2p^2 = (np + 2mq)^2 + 2(nq - mp)^2$.

Since $3 = 1 + 2 \cdot 1$ then $a = 3\left(p^2 + 2q^2\right) = (p + 2q)^2 + 2(q - p)^2$.

Case 2. Let $x \equiv \delta \pmod 3, y \equiv \sigma \pmod 3$, that is $x = 3p + \delta, y = 3q + \sigma$ where $|\sigma| = |\delta| = 1$.Then $3a = x^2 + 2y^2 \iff 3a = (3p + \delta)^2 + 2(3q + \sigma)^2 =$

$9p^2+6p\delta+18q^2+12q\sigma+2\sigma^2+\delta^2=9p^2+6p\delta+18q^2+12q\sigma+3 \implies$
$a=3p^2+6q^2+2(p\delta+2q\sigma)+1=3p^2\delta^2+6q^2\sigma^2+2(p\delta+2q\sigma)+1=$
$p^2\delta^2+4q^2\sigma^2+4pq\delta\sigma+2(p\delta+2q\sigma)+1+2p^2\delta^2-4pq\delta\sigma+2q^2\sigma^2=$
$(p\delta+2q\sigma)^2+2(p\delta+2q\sigma)+1+2(p\delta-2q\sigma)^2=(p\delta+2q\sigma+1)^2+2(p\delta-2q\sigma)^2$.

Problem 2.10

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be irredusible fractions for which equation $y=x^2+\dfrac{a}{b}x+\dfrac{c}{d}$ in integer x,y is solvable in integer x,y. Since $\dfrac{bc}{d}=by-bx^2-ax\in\mathbb{Z}$ and $\gcd(c,d)=1$ then $b\ \vdots\ d$, that is $b=kd$.

Since, $(y-x^2)d-c=\dfrac{adx}{b}=\dfrac{adx}{kd}=\dfrac{ax}{k}$ and $\gcd(a,b)=1 \implies \gcd(a,k)=1$ then $x=kt$ for some integer t.Hence,

$$y=x^2+\frac{a}{b}x+\frac{c}{d} \iff y=k^2t^2+\frac{akt}{kd}+\frac{c}{d} \iff y-k^2t^2=\frac{at+c}{d}.$$

Since $\gcd(c,d)=1$ and $\dfrac{at+c}{d}\in\mathbb{Z}$ yield $\gcd(a,d)=1$ (because otherwice,if $\gcd(a,d)=p\neq 1$) then $\dfrac{at+c}{d}\in\mathbb{Z} \implies \dfrac{at+c}{p}\in\mathbb{Z} \implies \dfrac{c}{p}\in\mathbb{Z} \iff \gcd(c,p)=p\neq 1$.

But $\gcd(c,d)=1 \implies \gcd(c,p)=1$.That is contradiction.

Thus, we obtain the following necessity condition for irreducible fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$:

1. $b=kd$, for some integer k;
2. $\gcd(a,d)=1$.

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be irredusible fractions such that $b=kd$, for some integer k and $\gcd(a,d)=1$.

Then equation $at+c\equiv 0 \pmod d$ have infinitely many solutions in integer t because $\gcd(a,d)=1$.

Let t be any such solution.Then for $x=kt$ we have

$$x^2+\frac{a}{b}x+\frac{c}{d}=k^2t^2+\frac{akt}{kd}+\frac{c}{d}=k^2t^2+\frac{at+c}{d}\in\mathbb{Z}.$$

★Problem 2.11(3932 CRUX)

Let x,y be natural solution of equation $x^2-14xy+y^2-4x=0$.

Since $x\equiv y \pmod 2$ and

$$x^2-14xy+y^2-4x=0 \iff y^2-14xy+49x^2=4\left(12x^2+x\right) \iff (y-7x)^2=2^2\left(12x^2+x\right)$$

then $12x^2+x=\left(\dfrac{y-7x}{2}\right)^2$, that is$12x^2+x$ is the perfect square of integer number.

Therefore, pair $(u,v)=\left(x,\left|\dfrac{y-7x}{2}\right|\right)$ is positive integer solution of equation $12u^2+u=v^2$. Opposite, let (x,z) is positive integer solution of equation $12x^2+x=z^2$.

Then x and $y=7x\pm 2z$ satisfy to equation $x^2-14xy+y^2-4x=0$.

Solving equation $12x^2+x=z^2$ in natural numbers.

Let $k:=\gcd(z,x)$ and let $a:=\dfrac{z}{k}, b:=\dfrac{x}{k}$ then $12x^2+x=z^2 \iff \dfrac{12x+1}{z}=\dfrac{z}{x}=\dfrac{a}{b} \iff$

$$\begin{cases} ax-bz=0 \\ 12bx-az=-b \end{cases} \iff \begin{cases} x=\dfrac{b^2}{a^2-12b^2} \\ z=\dfrac{ab}{a^2-12b^2} \end{cases}, \text{ where } a,b$$

Since $\gcd\left(a^2-12b^2,b^2\right)=\gcd\left(a^2,b^2\right)=1$ then x can be integer only if

$$\left|a^2-12b^2\right|=\gcd\left(a^2-12b^2,b^2\right)=1.$$

But since a^2+1 for any integer a isn't divisible by 3 then equation $a^2-12b^2=-1 \iff a^2+1=12b^2$ have no integer solutions. Thus, remains only equation $a^2-12b^2=1$. Since $a^2-12b^2=1$ implies that a,b are relatively prime then any natural solution (a,b) of Pell equation $a^2-12b^2=1$ induce natural solution $(x,z)=\left(b^2,ab\right)$ of equation $12x^2+x=z^2$,that is

$$S:=\left\{(x,z)\mid x,z\in\mathbb{N} \text{ and } 12x^2+x=z^2\right\}=\left\{\left(b^2,ab\right)\mid a,b\in\mathbb{N} \text{ and } a^2-12b^2=1\right\}$$

.("Pell parametrization" of all natural solutions of equation $12x^2+x=z^2$).

Note that $(a,b)=(7,2)$ is smallest natural solution of $a^2-12b^2=1$ and, therefore,

$$\left\{(a,b)\mid a,b\in\mathbb{N} \text{ and } a^2-12b^2=1\right\}=\{(a_n,b_n)\}_{n\geq 1}\ ,$$

where both a_n and b_n satisfy to the same recurrence $r_{n+2}-14r_{n+1}+r_n=0, n\in\mathbb{N}$ with different initial conditions $a_1=7, a_2=97, b_1=2, b_2=28$.Let $\delta_n:=7b_n-2a_n, n\in\mathbb{N}$ then $\delta_{n+2}-14\delta_{n+1}+\delta_n=0, n\in\mathbb{N}$ and $\delta_1=0, \delta_2=7\cdot 28-2\cdot 97=2$.

Since $\delta_2=2, \delta_2-\delta_1>0$ and $\delta_{n+2}-\delta_{n+1}=\delta_{n+1}-\delta_n+12\delta_n$ then by Math Induction $\delta_n>0$ for any $n>1$ and $\delta_{n+1}>\delta_n$ for any $n\in\mathbb{N}$.

Hence $7b_1^2-2a_1b_1=0$ and $7b_n^2-2a_nb_n=b_n\delta_n>0$ for any $n>1$.

So, all pairs $(x,y)=\left(b^2,7b^2\pm 2ab\right)$ where (a,b) be any solution of Pell equation $a^2-12b^2=1$ in natural numbers, excluding solution, indused by $(a,b)=(2,7)$ and $y=7b^2-2ab$, represent all natural solutions of equation $x^2-14xy+y^2-4x=0$.

Then $\gcd(x,y)=\gcd\left(b^2,7b^2\pm 2ab\right)=b\gcd(b,7b\pm 2a)=b\gcd(b,2a)=b\gcd(b,2)=2b$(because all b_n are even) and, therefore, $\gcd^2(x,y)=4x^2$.

Problem 2.12

So, we have to solve equation $16x+17y+21z=185,\ x,y,z\in\mathbb{N}\cup\{0\}$.

Let $u:=x+y, n:=185-21z$ then equation becames $16u+y=n$,where $u\geq y\geq 0$ and $0\leq z\leq 8$ because $185-21z\geq 0 \iff z\leq 8$.Let $t:=8-z$.Then $n=17+21t, 0\leq t\leq 8, z=8-t$.Since $y=n-16u$ then $0\leq y\leq u \iff 0\leq n-16u\leq u \iff \left\lceil\dfrac{n+16}{17}\right\rceil\leq u\leq\left\lfloor\dfrac{n}{16}\right\rfloor$.

Noting that $n\leq 185 \implies n<256 \iff \dfrac{n+16}{17}>\dfrac{n}{16}$ we conclude that equation $16u+y=n$ have integer solution (u,v) such that $u\geq y\geq 0$ iff $\left\lceil\dfrac{n+16}{17}\right\rceil=\left\lfloor\dfrac{n}{16}\right\rfloor$, that is iff for $t\in\{0,1,...,8\}$ holds

(1)
$$\left\lceil\frac{33+21t}{17}\right\rceil=\left\lfloor\frac{17+21t}{16}\right\rfloor.$$

Then for each such t we have $n=17+21t, u=\left\lfloor\dfrac{n}{16}\right\rfloor, y=16\left\{\dfrac{n}{16}\right\}, x=u-y, z=8-t$.

t	n	$\left\lceil\frac{n+16}{17}\right\rceil$	$\left\lfloor\frac{n}{16}\right\rfloor$	u	y	x	z
0	17	1	1	1	1	0	8
1	38	3	2				
2	59	4	3				
3	80	5	5	5	0	5	5
4	101	6	6	6	5	1	4
5	122	8	7				
6	143	9	8				
7	164	10	10	10	4	6	1
8	185	11	11	11	9	2	0

Thus, minimal number of boxes provide solution $(x,y,z)=(0,1,8)$.

Remark.

Intuitively, it is clear that the larger the capacity of the boxes involved in terms of transportation

for a given volume of cargo, the smaller number of boxes needed.

Hereof, maximal value of z which provide solvability of equation $16x+17y+21z=185$

in nonnegative integer x,y at the same time provide mimimal value of correspondent

sum $x+y+z$. For $z=8$ we we have $16x+17y+21\cdot 8=185 \iff 16x+17y=17 \iff \begin{cases} x=0\\ y=1\end{cases}$.

So, solution $(x,y,z)=(0,1,8)$ provide minimal number of boxes which is 9.

This intuitive and plausible reasoning leads to the following elegant solution:

Solution 2.

Let $t := x+y+z$.So, we should minimise t for nonnegative integer x, y, z that satisfy

$16x + 17y + 21z = 185.$

Note that $185 - 21z = 16x + 17y \geq 0 \implies 185 - 21z \geq 0 \iff z \leq \dfrac{185}{21} \overset{z\in\mathbb{Z}}{\iff} z \leq \left[\dfrac{185}{21}\right] = 8.$

Also note that $185 = 16x + 17y + 21z = 21t - 4y - 5x \implies 21t = 185 + 4y + 5x \geq 185 \implies$

$t \geq \dfrac{185}{21} \overset{z\in\mathbb{Z}}{\iff} t \geq \left[\dfrac{185+20}{21}\right] = 9$.From the other hand for $t = 9$ we obtain system of equations

$$\textbf{(2)}\quad \begin{cases} 16x + 17y + 21z = 185 \\ x + y + z = 9 \end{cases} \iff \begin{cases} y + 5z = 185 - 16(x+y+z) \\ x + y + z = 9 \end{cases} \iff$$

$$\begin{cases} y + 5z = 185 - 16\cdot 9 \\ x + y + z = 9 \end{cases} \iff \begin{cases} y + 5z = 41 \\ x = 9 - (y+z) \end{cases}$$

.

Since $x \geq 0 \iff y + z \leq 9$ and $y = 41 - 5z$ then $(41 - 5z) + z \leq 9 \iff 4z \geq 32 \iff z \geq 8.$

Thus, $8 \leq z \leq 8 \iff z = 8 \implies y = 1 \implies x = 0.$

Problem 2.13

Since $z = 10n - 5x - 2y$ and $z \geq 0$ then number of non-negative integer solutions of equation $5x + 2y + z = 10n$ equal to the number of non-negative integer solutions of inequality $5x + 2y \leq 10n$.

Then $\begin{cases} 5x + 2y \leq 10n \\ x, y \geq 0 \end{cases} \iff \begin{cases} 0 \leq y \leq \left[\dfrac{10n - 5x}{2}\right] \\ 0 \leq 10n - 5x, x \geq 0 \end{cases} \iff \begin{cases} 0 \leq x \leq 2n \\ 0 \leq y \leq \left[\dfrac{5(2n-x)}{2}\right] \end{cases}$.

Since for each $x \in \{0, 1, ..., 2n\}$ number of pairs (x, y) equal to $\left[\dfrac{5(2n-x)}{2}\right] + 1$ then number of non-negative integer solutions of equation $5x + 2y + z = 10n$ equal to the sum

$$\sum_{x=0}^{2n}\left(\left[\frac{5(2n-x)}{2}\right] + 1\right) = 2n + 1 + \sum_{k=0}^{2n}\left[\frac{5k}{2}\right] = 2n + 1 + \sum_{k=0}^{2n}\left(\left[\frac{k}{2}\right] + 2k\right) =$$

$$2n + 1 + \sum_{k=0}^{2n}\left[\frac{k}{2}\right] + \sum_{k=0}^{2n} 2k = 2n + 1 + 2n(2n+1) + \sum_{k=0}^{2n}\left[\frac{k}{2}\right] =$$

$$4n^2 + 2n + 1 + \sum_{i=1}^{n}\left[\frac{2i}{2}\right] + \sum_{i=1}^{n}\left[\frac{2i-1}{2}\right] = 4n^2 + 2n + 1 + \sum_{i=1}^{n} i + \sum_{i=1}^{n}(i-1) =$$

$$4n^2 + 2n + 1 + \sum_{i=1}^{n}(2i-1) = 4n^2 + 2n + 1 + \sum_{i=1}^{n}\left(i^2 - (i-1)^2\right) =$$

$$4n^2+2n+1+n^2=5n^2+2n+1.$$

3. Integer and fractional parts.

Problem 3.1
By Power Mean Inequality we have

$$\left(\frac{\sqrt[3]{2}+\sqrt[3]{4}}{2}\right)^3<\frac{2+4}{2}\iff\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3<24.$$

We try to prove that $\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3>23$. Since $\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3=6+6\left(\sqrt[3]{2}+\sqrt[3]{4}\right)$ then $\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3>23\iff 6\left(\sqrt[3]{2}+\sqrt[3]{4}\right)>17\iff 12\sqrt[3]{2}+12\sqrt[3]{4}>34$.

Note that $12\sqrt[3]{2}=3\sqrt[3]{2^7}=3\sqrt[3]{128}>3\sqrt[3]{125}=15$. Thus remains to prove $12\sqrt[3]{4}>19\iff 3^3\cdot 2^8>19^3$. We have $19^3=(20-1)^3=8000-1200+60-1=6859$ and $3^3\cdot 2^8=9\cdot 256\cdot 3=9\cdot 768=7680-768=6912>6859$.

Another proof of inequality $\sqrt[3]{2}+\sqrt[3]{4}>\dfrac{17}{6}$:

Let $x:=\sqrt[3]{2}+\sqrt[3]{4}$ then $\left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3=6+6\left(\sqrt[3]{2}+\sqrt[3]{4}\right)\iff x^3=6x+6$. So, $x>2$ is root of cubic equation $x^3-6x-6=0$. Let $p(x):=x^3-6x-6$.

Note that $p(x)\uparrow[\sqrt{2},\infty)$.Indeed, let $\sqrt{2}\le x_1<x_2$.Then $p(x_2)-p(x_1)=x_2^3-x_1^3-6(x_2-x_1)=(x_2-x_1)\left(x_2^2+x_2x_1+x_1^2-6\right)>0$,because $x_1^2\ge 2, x_2x_1>2, x_2^2>2$. Also, note then $p\left(\dfrac{17}{6}\right)=\dfrac{17^3}{6^3}-23<0$. Since $\dfrac{17}{6}>\sqrt{2}$ and $p(x)\uparrow[\sqrt{2},\infty)$ then $x>\dfrac{17}{6}$ (From supposition $\sqrt{2}<2<x<\dfrac{17}{6}$ follows $0=p(x)<p\left(\dfrac{17}{6}\right)$,that is the contradiction).

Problem-3.2
a) We have

$$\left[\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\right]=\left[3n+3+2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)}\right]=$$

$$3n+3+\left[2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)}\right].$$

Since by 2-AM-GM Inequality $2\sqrt{n(n+1)}<n+(n+1)=2n+1, 2\sqrt{n(n+2)}<n+(n+2)=2n+2, 2\sqrt{(n+1)(n+2)}<(n+1)+(n+2)=2n+3$ (we wrote $<$ instead $\le$ because condition of equality in AM-GM Inequality is not fulfilled) we obtain

$$2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)}<6n+6$$

.

(Or, since $2\sqrt{n(n+1)} = \sqrt{4n^2+4n} < \sqrt{4n^2+4n+1} = 2n+1$, $2\sqrt{n(n+2)} = \sqrt{4n^2+8n} < \sqrt{4n^2+8n+4} = 2n+1$ and $2\sqrt{(n+1)(n+2)} = \sqrt{4n^2+6n+8} < \sqrt{4n^2+12n+9} = 2n+3$ then $2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)} < 6n+6$)

We will find lower bound for $2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)}$ using 3-AM-GM Inequality again, namely,

$$\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+2)} > 3\sqrt[3]{\sqrt{n(n+1)}\cdot\sqrt{n(n+2)}\cdot\sqrt{(n+1)(n+2)}} =$$

$$3\sqrt[6]{n(n+1)\cdot n(n+2)\cdot(n+1)(n+2)} = 3\sqrt[3]{n(n+1)(n+2)}.$$

Note that $n(n+1)(n+2) = n^3+3n^2+2n \geq (n+5/6)^3$ for any $n \geq 2$. Indeed,

$$n^3+3n^2+2n-(n+5/6)^3 = \frac{n^2}{2} - \frac{n}{12} - \frac{125}{216} = \frac{n}{12}(6n-1) - \frac{125}{216} \geq$$

$$\frac{n}{12}(6\cdot 2-1) - \frac{125}{216} \geq \frac{11}{12}\cdot 2 - \frac{125}{216} = \frac{11}{6} - \frac{125}{216} > 0.$$

Hence,

$$2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)} > 2\cdot 2(n+5/6) = 6n+5.$$

Since

$$6n+5 < 2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)} < 6n+6$$

then

$$\left[2\sqrt{n(n+1)}+2\sqrt{n(n+2)}+2\sqrt{(n+1)(n+2)}\right] = 6n+5$$

and, therefore,

$$\left[\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\right] = 9n+8.$$

b) Since* $[\sqrt{x}] = \left[\sqrt{[x]}\right]$ then, using **a)** we obtain

$$\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right] = \left[\sqrt{\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2}\right] =$$

$$\left[\sqrt{\left[\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\right]}\right] = \left[\sqrt{9n+8}\right].$$

Appendix.

Let $p := [\sqrt{x}]$ then$p \geq 0$ and $p \leq \sqrt{x} < p+1 \iff p^2 \leq x < (p+1)^2$ .Since p^2 is integer lower bound for x and $[x]$ is biggest integer lower bound for x then $p^2 \leq [x] \leq x < (p+1)^2 \implies p^2 \leq [x] < (p+1)^2 \implies p \leq \sqrt{[x]} < p+1 \iff \left[\sqrt{[x]}\right] = p$.

Problem 3.3

Obviously that $x \neq 0$ and easy to see that $x \notin \mathbb{Z}$ because otherwice $\left\{\dfrac{1}{x}\right\} = 1$, i.e. contradiction. Denoting $n := [x] + \left[\dfrac{1}{x}\right] + 1$ we obtain that if x is solution of equation $\{x\} + \left\{\dfrac{1}{x}\right\} = 1$ then $x \notin \mathbb{Z}$ and for some integer n this x satisfy equation $x + \dfrac{1}{x} = n$. Moreover, since $|x| \neq 1$ then $|n| = \left|x + \dfrac{1}{x}\right| = |x| + \dfrac{1}{|x|} > 2$.

Let now n is integer such that $|n| > 2$ then equation $x + \dfrac{1}{x} = n \iff x^2 - nx + 1 = 0$ have two irrational solutions $x = \dfrac{n + \sqrt{n^2-4}}{2}$ or $x = \dfrac{n - \sqrt{n^2-4}}{2} =$ and for each we have

$$\{x\} + \left\{\frac{1}{x}\right\} = \{x\} + \{n - x\} = \{x\} + \{-x\} = 1.$$

Thus all solution of equation $\{x\} + \left\{\dfrac{1}{x}\right\} = 1$ can be represented in form $x = \dfrac{n \pm \sqrt{n^2-4}}{2}$, where $n \in \mathbb{Z}$ and $|n| > 2$.

◊

Problem 3.4.

Let $S_n := \left\{(a,b) \mid a,b \in \mathbb{N} \setminus \{1\} \text{ and } a^b \leq n\right\}$. Note that $a^b \leq n \iff a \leq \sqrt[b]{n} \iff a \leq [\sqrt[b]{n}]$. For any $b \in \{2,3,...,n\}$ let $A_b := \{a \mid a \in \mathbb{N} \setminus \{1\} \text{ and } a \leq [\sqrt[b]{n}]\}$. Then $S_n = \bigcup\limits_{b=2}^{n} A_b \times \{b\}$ and, therefore, $|S_n| = \sum\limits_{b=2}^{n} |A_b| = \sum\limits_{b=2}^{n} ([\sqrt[b]{n}] - 1) = \sum\limits_{b=2}^{n} [\sqrt[b]{n}] - (n-1)$.

Note that $a^b \leq n \iff b \leq \log_a n \iff b \leq [\log_a n]$. For any $a \in \{2,3,...,n\}$ let $B_a := \{b \mid b \in \mathbb{N} \setminus \{1\} \text{ and } b \leq [\log_a n]\}$ .Then $S_n = \bigcup\limits_{b=2}^{n} \{a\} \times B_a$ and, therefore, $|S_n| = \sum\limits_{a=2}^{n} |B_a| = \sum\limits_{b=2}^{n} ([\log_a n] - 1) = \sum\limits_{b=2}^{n} [[\log_a n]] - (n-1)$ .Thus, $\sum\limits_{b=2}^{n} [[\log_a n]] - (n-1) = \sum\limits_{b=2}^{n} [\sqrt[b]{n}] - (n-1) \iff \sum\limits_{a=2}^{n} [\log_a n] = \sum\limits_{b=2}^{n} [\sqrt[b]{n}]$.

★Problem 3.5 (CRUX #3095)

Consider two cases:

1. Let $\left[\dfrac{c+pb}{q}\right] \leq a$, then $p\left[\dfrac{c+pb}{q}\right] \leq pa$ and we have inequality

$$\left[\frac{c+pb}{q}\right] \leq \frac{c+pb}{q} \iff c+pb-q\left[\frac{c+pb}{q}\right] \geq 0.$$

Thus

$$\frac{c+p\,(a+b)}{p+q} - \left[\frac{c+pb}{q}\right] = \frac{c+pa+pb-(p+q)\left[\frac{c+pb}{q}\right]}{p+q} =$$

$$\frac{\left(pa-p\left[\frac{c+pb}{q}\right]\right)+\left(c+pb-q\left[\frac{c+pb}{q}\right]\right)}{p+q} \geq 0$$

.

So, we obtain inequality $\left[\frac{c+pb}{q}\right] \leq \frac{c+p\,(a+b)}{p+q}$, which implies $\left[\frac{c+pb}{q}\right] \leq \left[\frac{c+p\,(a+b)}{p+q}\right]$.

2\. Let $\left[\frac{c+pb}{q}\right] \geq a$, then $c+pb-qa \geq 0$ and consequently

$$\left[\frac{c+p\,(a+b)}{p+q}\right] = \left[\frac{c+pb-qa+a\,(p+q)}{p+q}\right] = a + \left[\frac{c+pb-qa}{p+q}\right] \geq a.$$

Problem 3.6

a. Let $p := \left[n\sqrt{2}\right]$ then $p < n\sqrt{2} < p+1$($n\sqrt{2} \neq p$ because $\sqrt{2} \notin \mathbb{Q}$) and

$$\left\{n\sqrt{2}\right\} = n\sqrt{2}-p = \frac{2n^2-p^2}{n\sqrt{2}+p} > \frac{2n^2-p^2}{2n\sqrt{2}} \geq \frac{1}{2n\sqrt{2}}$$

because $p < n\sqrt{2} \implies p^2 < 2n^2 \iff 1 \leq 2n^2 - p^2$

b. We will consider now natural n such that $2n^2-1$ be a perfect square (for example $n = 1, 5$) that is we will find all pairs (n,p) of natural numbers such that $2n^2-p^2 = 1$.

Let (n,p) be such pair. Then $2n^2 = p^2+1 < (p+1)^2$ and, therefore, $p^2 < 2n^2 < (p+1)^2 \iff p < n\sqrt{2} < p+1 \iff \left[n\sqrt{2}\right] = p$.

Also, since $1 = \left(\sqrt{2}+1\right)^2\left(\sqrt{2}-1\right)^2 = \left(3+2\sqrt{2}\right)\left(3-2\sqrt{2}\right)$ then $1 = 2n^2 - p^2 = \left(n\sqrt{2}+p\right)\left(n\sqrt{2}-p\right) = \left(n\sqrt{2}+p\right)\left(3+2\sqrt{2}\right)\left(n\sqrt{2}-p\right)\left(3-2\sqrt{2}\right) = \left((3n+2p)\sqrt{2}+4n+3p\right)\left((3n+2p\right.$ $2\,(3n+2p)^2-(4n+3p)^2$.

Thus, $(3n+2p, 4n+3p)$ is natural solution of equation $2x^2-y^2 = 1$ and starting from solution $(1,1)$ we obtain infinite sequence $(n_1,p_1), (n_2,p_2), ..., (n_k,p_k), ...$of natural solutions of equation $2x^2-y^2=1$ defined recursively as follows

$$\textbf{(1)} \qquad \begin{cases} n_{k+1} = 3n_k + 2p_k \\ p_{k+1} = 4n_k + 3p_k \end{cases}, k \in \mathbb{N}\ and n_1 = p_1 = 1.$$

Since $n_{k+1} > 3n_k \iff \dfrac{n_{k+1}}{3^{k+1}} > \dfrac{n_k}{3^k}, k \in \mathbb{N}$ and then $\dfrac{n_{k+1}}{3^{k+1}} > \dfrac{n_1}{3^1} = \dfrac{1}{3} \implies n_{k+1} > 3^k, k \in \mathbb{N} \implies n_k \geq 3^{k-1}, k \in \mathbb{N}$.(Similarly $p_k \geq 3^{k-1}$).

Recall that $p_k = \left[n_k\sqrt{2}\right]$ .Then $\left\{n_k\sqrt{2}\right\} = n_k\sqrt{2} - p_k = \dfrac{1}{n_k\sqrt{2}+p_k}$ and

$$\frac{1}{n_k\sqrt{2}+p_k} - \frac{1}{2n_k\sqrt{2}} = \frac{n_k\sqrt{2}-p_k}{2n_k\sqrt{2}\left(n_k\sqrt{2}+p_k\right)} = \frac{1}{2n_k\sqrt{2}\left(n_k\sqrt{2}+p_k\right)^2} <$$

$$\frac{1}{2n_k\sqrt{2}\left(3^{k-1}\sqrt{2}+3^{k-1}\right)^2} = \frac{1}{2n_k\sqrt{2}} \cdot \frac{1}{3^{2(k-1)}\left(2\sqrt{2}+3\right)} < \frac{1}{2n_k\sqrt{2}} \cdot \frac{1}{3^{2k-1}}.$$

Since for any positive real ε we can find k such that $\dfrac{1}{3^{2k-1}} < \varepsilon$ (for example take any $k > \log_9 \dfrac{1}{3\varepsilon}$) then for this k we have

$$\frac{1}{n_k\sqrt{2}+p_k} - \frac{1}{2n_k\sqrt{2}} < \frac{1}{2n_k\sqrt{2}} \cdot \varepsilon \iff \left\{n_k\sqrt{2}\right\} < \frac{1+\varepsilon}{2n_k\sqrt{2}}.$$

Remark.

Note that since $2p_{k+1} = 8n_k + 6p_k$ and $2p_k = n_{k+1} - 3n_k$ we have $n_{k+2} - 3n_{k+1} = 8n_k + 3\left(n_{k+1} - 3n_k\right) \iff n_{k+2} - 6n_{k+1} + n_k = 0, k \in \mathbb{N}$ and $n_1 = 1, n_2 = 5$.Also note that $2n_k^2 - \left[n_k\sqrt{2}\right]^2 = 1$ and $\left\{n_k\sqrt{2}\right\} < \dfrac{1+1/3^{2k-1}}{2n_k\sqrt{2}}$. ♢

Problem 3.7

Let $p := \left[\sqrt[4]{n}\right]$. Since $n \in \mathbb{N}$ isn't forth degree of natural number then $p < \sqrt[4]{n} < p+1 \iff p^4 < n < (p+1)^4$ and

$$\left\{\sqrt[4]{n}\right\} = \sqrt[4]{n} - p = \frac{n-p^4}{\sqrt[4]{n^3}+p\sqrt[4]{n^2}+p^2\sqrt[4]{n}+p^3} \geq \frac{1}{\sqrt[4]{n^3}+p\sqrt[4]{n^2}+p^2\sqrt[4]{n}+p^3} > \frac{1}{4\sqrt[4]{n^3}}$$

because $\sqrt[4]{n^3} + p\sqrt[4]{n^2} + p^2\sqrt[4]{n} + p^3 < \sqrt[4]{n^3} + \sqrt[4]{n} \cdot \sqrt[4]{n^2} + \left(\sqrt[4]{n}\right)^2 \sqrt[4]{n} + \left(\sqrt[4]{n}\right)^3$.

♢

★Problem -3.8. (J289, MR) (Identity with integer parts).

Let $n := \left[\dfrac{1}{1-a}\right]$ then $1 \leq n \leq \dfrac{1}{1-a} < n+1 \implies \dfrac{1}{n+1} < 1-a \leq \dfrac{1}{n} \iff \dfrac{n-1}{n} \leq a < \dfrac{n}{n+1}$ and, therefore, $\dfrac{(n-1)(n+1)}{n} \leq a\left(1+\left[\dfrac{1}{1-a}\right]\right) < \dfrac{n(n+1)}{n+1} \iff \dfrac{n^2-1}{n} + 1 \leq a\left(1+\left[\dfrac{1}{1-a}\right]\right) + 1 < n+1$.Since $\dfrac{n^2-1}{n} + 1 = n - \dfrac{1}{n} + 1 \geq n$ for any natural n then

$$n \leq a\left(1+\left[\frac{1}{1-a}\right]\right) + 1 < n+1 \iff \left[a\left(1+\left[\frac{1}{1-a}\right]\right)\right] + 1 = n.$$

Thus, $\left[a\left(1+\left[\dfrac{1}{1-a}\right]\right)\right]+1=\left[\dfrac{1}{1-a}\right]$.

Or, such variant:

Prove that $\left[\left(1-\dfrac{1}{x}\right)(1+[x])\right]+1=[x]$ for any real $x\geq 1$. Let $n:=[x]$ then $1\leq n\leq x<n+1 \implies \dfrac{1}{n+1}<\dfrac{1}{x}\leq\dfrac{1}{n} \iff \dfrac{n-1}{n}\leq 1-\dfrac{1}{x}<\dfrac{n}{n+1}$ and, therefore,

$$\frac{(n-1)(n+1)}{n}\leq\left(1-\frac{1}{x}\right)(1+[x])<\frac{n(n+1)}{n+1} \iff \frac{n^2-1}{n}+1\leq\left(1-\frac{1}{x}\right)(1+[x])+1<n+1.$$

Since $\dfrac{n^2-1}{n}+1=n-\dfrac{1}{n}+1\geq n$ for any natural n then

$$n\leq\left(1-\frac{1}{x}\right)(1+[x])+1<n+1 \iff \left[\left(1-\frac{1}{x}\right)(1+[x])\right]+1=n.$$

Thus, $\left[\left(1-\dfrac{1}{x}\right)(1+[x])\right]+1=[x]$.

Problem 3.9.

Let $a_n:=\left(m+\sqrt{m^2-1}\right)^n+\left(m-\sqrt{m^2-1}\right)^n$ then for a_n we have recurrence

$a_{n+1}-2ma_n+a_{n-1}=0$ and initial conditions $a_0=2, a_1=2m$.

So, a_n is integer and even for all n and since, $m-\sqrt{m^2-1}\in(0,1)$ then $\left(m-\sqrt{m^2-1}\right)^n\in(0,1)$ and sequently $1-\left(m-\sqrt{m^2-1}\right)^n\in(0,1)$.

Therefore in representation

$\left(m+\sqrt{m^2-1}\right)^n=a_n-\left(m-\sqrt{m^2-1}\right)^n=(a_n-1)+\left(1-\left(m-\sqrt{m^2-1}\right)^n\right)$

$\left\lfloor\left(m+\sqrt{m^2-1}\right)^n\right\rfloor=a_n-1$ and $\left\{\left(m+\sqrt{m^2-1}\right)^n\right\}=1-\left(m-\sqrt{m^2-1}\right)^n$.

Thus, $\left\lfloor\left(m+\sqrt{m^2-1}\right)^n\right\rfloor$ is odd for all n.

★ Poblem 3.10 (W 16, J.Wildt IMO 2017)

For given natural $n>1$ let $I_n:=\{1,2,...,n\}$ and let $f(k):=\left[\dfrac{k^2}{n}\right]$ for any $k\in I_n$.

Then we should determine $|f(I_n)|$.

Consider two cases.

Case1. n is even, that is $n=2m$.

Lemma.

For any $k\in I_m$ holds inequality $f(k+1)-f(k)\leq 1$.

Proof.

First we consider $k\in I_{m-1}$. Let $1\leq k\leq m-1$ then $f(k+1)=\left[\dfrac{k^2+2k+1}{2m}\right]$ and $1+f(k)=\left[\dfrac{k^2+2m}{2m}\right]$.

Note that $k \leq m-1$ yields

$$f(k+1) \leq \left[\frac{k^2+2(m-1)+1}{2m}\right] = \left[\frac{k^2+2m-1}{2m}\right] \leq \left[\frac{k^2+2m}{2m}\right] = 1+f(k).$$

Also for $k=m$ we have

$$f(m+1) = \left[\frac{m^2+2m+1}{2m}\right] = \left[\frac{\frac{m^2+2m+1}{2}}{m}\right] =$$

$$\left[\frac{\left[\frac{m^2+2m+1}{m}\right]}{2}\right] = \left[\frac{m+2+\left[\frac{1}{m}\right]}{2}\right] = 1+\left[\frac{m}{2}\right] = 1+f(m).$$

Corollary.

$$f(I_m) = \left\{0,1,2,...,\left[\frac{m}{2}\right]\right\}$$

.

Proof.

Note that $f(k)$ isn't decreasing, that is $f(k+1) - f(k) = \left[\frac{(k+1)^2}{n}\right] - \left[\frac{k^2}{n}\right] \geq 0$. Also, $f(m) = \left[\frac{m^2}{2m}\right] = \left[\frac{m}{2}\right]$ and $f(1) = \left[\frac{1}{2m}\right] = 0$. Suppose that there is $i \in I_{m/2}$ for which $f^{-1}(i) = \varnothing$.Obvious that $1 \leq i < \left[\frac{m}{2}\right]$.Let $k_* := \{k \mid k \in I_m \text{ and } f(k) < i\}$. Then $f(k_*) < i < f(k_*+1) \implies f(k_*+1) - f(k_*) > 1$,that is contradiction to **Lemma.■**

Now note that $f(k)$ is strictly increasing in $k \in \{m+1, m+2, .., 2m\}$. Indeed, since for any $k \in I_m$ we have

$$f(m+k) = \left[\frac{(m+k)^2}{2m}\right] = \left[\frac{m^2+2mk+k^2}{2m}\right] = k+\left[\frac{m^2+k^2}{2m}\right]$$

then

$$f(m+(k+1)) = k+1+\left[\frac{m^2+(k+1)^2}{2m}\right] > k+\left[\frac{m^2+k^2}{2m}\right] = f(m+k) \text{ for any } k \in I_{m-1}$$

.

Hence, $|f(I_{2m} \diagdown I_m)| = m$ and since $|f(I_m)| = \left[\frac{m}{2}\right]+1$ then $|f(I_{2m})| = m+\left[\frac{m}{2}\right]+1 = \left[\frac{3m+2}{2}\right]$.

Case 2. n is odd, that is $n = 2m + 1$.
Then as above we will prove divide this case on two parts.
First we consider f on I_{m+1}.
For any $k \in I_m$ we have

$$f(k+1) = \left[\frac{k^2+2k+1}{2m+1}\right] \le \left[\frac{k^2+2m+1}{2m+1}\right] = 1 + \left[\frac{k^2}{2m+1}\right] = 1 + f(k+1)$$

and

$$f(m+1) = \left[\frac{m^2+2m+1}{2m+1}\right] = 1 + \left[\frac{m^2}{2m+1}\right],\ f(1) = \left[\frac{1}{2m+1}\right] = 0.$$

By the same way as above, using inequality $f(k+1) \le f(k) + 1$, can be proved that for any $0 < i < 1 + \left[\dfrac{m^2}{2m+1}\right]$ there is preimage in I_{m+1}.

So, $|f(I_{m+1})| = 1 + \left[\dfrac{m^2}{2m+1}\right]$.

Remains consider behavior of f on $I_{2m+1} \diagdown I_{m+1} = \{m+1+k \mid k \in I_m\}$.
For any $k \in I_m$ we have

$$f(m+1+k) = \left[\frac{(m+1)^2 + 2(m+1)k + k^2}{2m+1}\right] = k + \left[\frac{k^2+k+(m+1)^2}{2m+1}\right]$$

and then

$$f(m+1+(k+1)) \ge k+1+\left[\frac{(k+1)^2+(k+1)+(m+1)^2}{2m+1}\right] > k+\left[\frac{k^2+k+(m+1)^2}{2m+1}\right] = f(m+1+k).$$

Since $f(k)$ is strictly increasing in $k \in I_{2m+1} \diagdown I_{m+1}$ then $|f(I_{2m+1} \diagdown I_{m+1})| = m$.Thus,

$$|f(I_{2m+1})| = m + 1 + \left[\frac{m^2}{2m+1}\right] = 1 + \left[\frac{3m^2+m}{2m+1}\right].$$

$$\text{So, } |f(I_n)| = \begin{cases} 1 + \left[\dfrac{3m}{2}\right] & \text{if } n = 2m \\ 1 + \left[\dfrac{3m^2+3m+1}{2m+1}\right] & \text{if } n = 2m+1 \end{cases}.$$

For $n = 2m$ we have $|f(I_n)| = 1 + \left[\dfrac{3m}{2}\right] = 1 + \left[\dfrac{6m}{4}\right] = 1 + \left[\dfrac{3n}{4}\right]$.
For $n = 2m + 1$ we have

$$|f(I_n)| = 1 + \left[\frac{3m^2+3m+1}{2m+1}\right] = 1 + \left[\frac{\dfrac{12m^2+12m+4}{2m+1}}{4}\right] =$$

$$1+\left[\frac{\frac{3(2m+1)^2+1}{2m+1}}{4}\right]=1+\left[\frac{\left[3(2m+1)+\frac{1}{2m+1}\right]}{4}\right]=1+\left[\frac{3(2m+1)}{4}\right]=1+\left[\frac{3n}{4}\right].$$

So, $|f(I_n)|=1+\left[\frac{3n}{4}\right]$.

Problem 3.11(U182)

Let $x\in\left(\frac{1}{2},1\right)$.Define sequence $(x_n)_{n\geq 0}$ as follows:

$x_0:=x$ and $x_n=2x_{n-1}-1, n\geq 1$.

Then for any $x\in\left(\frac{1}{2},1\right)$ there is n such that $x_n\in\left[0,\frac{1}{2}\right]$. Indeed, from $x_{n+1}=2x_n-1 \iff x_{n+1}-1=2(x_n-1)$ follows $x_n-1=2^n(x_0-1) \iff x_n=1-2^n(1-x)$.

Since

$$0\leq x_n\leq\frac{1}{2} \iff 0\leq 1-2^n(1-x)\leq\frac{1}{2} \iff \begin{cases} 2^n\leq\frac{1}{1-x} \\ \frac{1}{1-x}\leq 2^{n+1} \end{cases} \iff$$

$$\log_2\frac{1}{1-x}-1\leq n\leq\log_2\frac{1}{1-x}$$

then $\left\lfloor\log_2\frac{1}{1-x}\right\rfloor$ is such n, (because for any real a by definition of integer part of a we have $\lfloor a\rfloor\leq a<\lfloor a\rfloor+1 \iff a-1<\lfloor a\rfloor\leq a$).

Hence, for such n we obtain $f(x)=f(x_0)=f(2x_0-1)=f(x_1)=f(2x_1-1)=f(x_2)=...=f(x_{n-1})=f(2x_{n-1}-1)=f(x_n)=c$. Thus $f(x)=c$ for any $x\in[0,1)$ and, since by condition $f(x)$ is continuous on $[0,1]$, then $f(1)=\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}c=c$.

◊

◊

4. Equations, systems of equations.

★Problem 4.1(Generalization of M703* Kvant**)**

Let $a:=q+r, b:=r+p, c:=p+q$.Since $p,q,r>0$ then a,b,c satisfy to triangle inequalities and, therefore, numbers a,b,c determiner a triangle ABC with sidelengths $a=BC, b=CA, c=AB$. Note that x,y,z have the same sign and since $xy+yz+zx$ and $(q+r)(x+1/x)=(r+p)(y+1/y)=(p+q)(z+1/z)$ are invariant with respect to transformation $(x,y,z)\longmapsto(-x,-y,-z)$ we further assume that $x,y,x>0$.

Let $\alpha := 2\tan^{-1} x, \beta := 2\tan^{-1} y, \gamma := 2\tan^{-1} z$. Since $x, y, z > 0$ then

$\alpha, \beta, \gamma \in (0, \pi)$, $x+1/x = \dfrac{2}{\sin\alpha}$, $y+1/y = \dfrac{2}{\sin\beta}$, $z+1/z = \dfrac{2}{\sin\gamma}$, $xy+yz+zx = 1 \iff$

$$\textbf{(1)} \qquad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$$

and

$$(q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z)$$

can be rewritten in the form

$$\textbf{(2)} \qquad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

.

Now we will pay attention to the correlation **(1).**

We have **(1)** $\iff \tan\dfrac{\alpha}{2}\left(\tan\dfrac{\beta}{2} + \tan\dfrac{\gamma}{2}\right) = 1 - \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$. Note that $\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} \neq 1$ because otherwise since $\beta, \gamma \in (0, \pi)$ we obtain $\tan\dfrac{\alpha}{2} = 0 \iff \alpha = 0$ (contradiction with $\alpha > 0$). Thus,

$$(1) \iff \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}}{1 - \tan\frac{\beta}{2}\tan\frac{\gamma}{2}} \iff \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \tan\left(\frac{\beta}{2} + \frac{\gamma}{2}\right) \iff$$

$\dfrac{\pi}{2} - \dfrac{\alpha}{2} = \dfrac{\beta}{2} + \dfrac{\gamma}{2} \iff \alpha + \beta + \gamma = \pi$ (because $\dfrac{\pi}{2} - \dfrac{\alpha}{2}, \dfrac{\beta}{2} + \dfrac{\gamma}{2} \in \left(0, \dfrac{\pi}{2}\right)$).

Since $\alpha, \beta, \gamma \in (0, \pi)$ and $\alpha + \beta + \gamma = \pi$ then α, β, γ can be considered as angles of some triangle with correspondent sidelengths $\sin\alpha, \sin\beta, \sin\gamma$ which due to **(2)** is similar to triangle ABC.Hence, $\alpha = A, \beta = B, \gamma = C$ and, therefore, $(x, y, z) = \left(\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}\right)$ and $(x, y, z) = \left(-\tan\dfrac{A}{2}, -\tan\dfrac{B}{2}, -\tan\dfrac{C}{2}\right)$ all solutions of the system

$$\begin{cases} a(x+1/x) = b(y+1/y) = c(z+1/z) \\ xy + yz + zx = 1 \end{cases}$$

.

It remains only to express $\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ via p, q, r. Let $s := \dfrac{a+b+c}{2} = p+q+r$ then

$$\tan\frac{A}{2} = \frac{r}{s-a} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{qr}{p(p+q+r)}}$$

and, cyclic,

$$\tan\frac{B}{2}=\sqrt{\frac{rp}{q(p+q+r)}},\tan C=\sqrt{\frac{pq}{r(p+q+r)}}.$$

So, $(x,y,z)=\pm\frac{1}{\sqrt{p+q+r}}\left(\sqrt{\frac{qr}{p}},\sqrt{\frac{rp}{q}},\sqrt{\frac{pq}{r}}\right)$ all solutions of original system.

Problem 4.2

First note that $x,y,z\neq\pm1$ (if, for example, $x^2=1$ then first equation give us $x=0$).

$$\text{Then}\begin{cases}2x+x^2y=y\\2y^2+y^2z=z\\2z^2+z^2x=x\end{cases}\iff\begin{cases}y=\dfrac{2x}{1-x^2}\\z=\dfrac{2y}{1-y^2}\\x=\dfrac{2z}{1-z^2}\end{cases}.$$

Let $\alpha:=\tan^{-1}(x)$ then $x=\tan\alpha$, where $\alpha\in(-\pi/2,\pi/2)$. Hence, $y=\tan 2\alpha,z=\tan 4\alpha$ and third equation becomes $\tan\alpha=\tan 8\alpha\iff 8\alpha=\alpha+k\pi\iff 7\alpha=k\pi\iff\alpha=\frac{k\pi}{7}$, where $k\in\mathbb{Z}$ and $\left|\frac{k\pi}{7}\right|<\frac{\pi}{2}\iff|k|\leq 3$. Thus, $(x,y,z)=\left(\tan\frac{k\pi}{7},\tan\frac{2k\pi}{7},\tan\frac{4k\pi}{7}\right),k\in\{-3,-2,-1,0,1,2,3\}$ represent all solutions of the system.

Problem 4.3

Let $f(x):=x-\sin x$.Then system becomes $\begin{cases}y=f(x)\\z=f(y)\\x=f(z)\end{cases}$.

Note that function $f(x)$ increasing in $\mathbb{R}$.Indeed, let $x_1<x_2$ and $0<x_2-x_1<\pi$ then $f(x_2)-f(x_1)=x_2-\sin x_2-(x_1-\sin x_1)=x_2-x_1-(\sin x_2-\sin x_1)=x_2-x_1-2\cos\frac{x_2+x_1}{2}\sin\frac{x_2-x_1}{2}\geq x_2-x_1-2\sin\frac{x_2-x_1}{2}>0$ because $\sin t<t$ for $0<t<\frac{\pi}{2}$.

Assume that $x\neq y$ let it be $x<y$ then $f(x)<f(y)\iff y<z\implies f(y)<f(z)\iff z<x$. Thus, $x<y<z<x$ that is the contradiction.

If $y<x$ then $f(y)<f(x)\iff z<y\implies f(z)<f(y)\iff x<z\implies f(x)<f(z)\iff y<x$,that is the contradiction again. So, $x=y$. Similarly we get $y=z$ and, therefore, $x=y=z=t$ where t is any solution of equation $\sin t=0$,that is $t=n\pi,n\in\mathbb{Z}$.

Problem 4.4.

First we will prove, using Math Induction, that for any real $x_1,x_2,...,x_n$ holds inequality

$$\textbf{(1)}\qquad n\left(x_1^2+x_2^2+...+x_n^2\right)\geq(x_1+x_2+...+x_n)^2$$

and equality occurs iff $x_1 = x_2 = ... = x_n$.

1. Base of Math Induction.

$2\left(x_1^2 + x_2^2\right) \geq (x_1 + x_2)^2 \iff (x_1 - x_2)^2 \geq 0$ and equality occurs iff $x_1 = x_2$.

2. Step of Math Iduction.

Since $(x_1 + x_2 + ... + x_n)^2 \leq n\left(x_1^2 + x_2^2 + ... + x_n^2\right)$ by supposition of Math Induction then

$$(x_1 + x_2 + ... + x_n + x_{n+1})^2 = (x_1 + x_2 + ... + x_n)^2 + 2x_{n+1}(x_1 + x_2 + ... + x_n) + x_{n+1}^2 \leq$$

$$n\left(x_1^2 + x_2^2 + ... + x_n^2\right) + 2x_{n+1}(x_1 + x_2 + ... + x_n) + x_{n+1}^2 + \left(x_{n+1}^2 + x_1^2\right) + \left(x_{n+1}^2 + x_2^2\right) + ... + \left(x_{n+1}^2 + x_n^2\right) =$$

$(n+1)\left(x_1^2 + x_2^2 + ... + x_n^2 + x_{n+1}^2\right)$ because $2x_{n+1}x_i \leq x_{n+1}^2 + x_i^2, i = 1, 2, ..., n$.

Equality occurs by supposition of Math Induction iff $x_1 = x_2 = ... = x_n$ and $x_{n+1} = x_i, i = 1, 2, ..., n$ by base of Math Induction.

Coming back to the system, since $n\left(x_1^2 + x_2^2 + ... + x_n^2\right) = (x_1 + x_2 + ... + x_n)^2$ we can conclude that $x_1 = x_2 = ... = x_n = \dfrac{1}{n}$.

Problem 4.5.

a) We have $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y + az = 1 + a \end{cases} \iff \begin{cases} x^2 + y^2 = 1 - z^2 \\ x + y = 1 + a - az \end{cases}$.

Since $(x - y)^2 = 2\left(x^2 + y^2\right) - (x + y)^2 = 2\left(1 - z^2\right) - (1 + a - az)^2 = 2\left(1 - z^2\right) - (1 + a - az)^2 = -\left(a^2 + 2\right)z^2 + 2a(a+1)z - a^2 - 2a + 1$ then $-\left(a^2 + 2\right)z^2 + 2a(a+1)z - a^2 - 2a + 1 \geq 0 \iff \left(a^2 + 2\right)z^2 - 2a(a+1)z + a^2 + 2a - 1 \leq 0$, where latter inequality solvable iff discriminant of quadratic trinomial isn't negative, that is iff $a^2(a+1)^2 - \left(a^2 + 2\right)\left(a^2 + 2a - 1\right) = 2 - 4a \geq 0 \iff a \leq 1/2$.

Thus, for $a \geq 1/2$ the system is solvable only if $a = 1/2$ and in that case we get for z inequality $\left((1/2)^2 + 2\right)z^2 - 3/2z + (1/2)^2 \leq 0 \iff \dfrac{1}{4}(3z - 1)^2 \leq 0 \iff z = 1/3$.

For $a = 1/2$ and such z the system becomes

$$\begin{cases} x^2 + y^2 = 8/9 \\ x + y = 4/3 \end{cases} \iff x = y = 2/3.$$

So solution is $(x, y, z) = (2/3, 2/3, 1/3)$.

b) Solution 1.

Let x, y, z be solution of the system. Then x, y, z can be represented as solutions of the cubic equation

$$(u - x)(u - y)(u - z) = 0 \iff u^3 - (x + y + z)u^2 + (xy + yz + zx)u - xyz = 0 \iff$$

$$\textbf{(1)} \qquad u^3 - au^2 + (xy + yz + zx)u - xyz = 0.$$

Since $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{a} \iff a(xy+yz+zx)=xyz$ then equation **(1)** becomes

$$u^3-au^2+(xy+yz+zx)u-a(xy+yz+zx)=0 \iff$$

$$\left(xy+xz+yz+u^2\right)(u-a)=0.$$

Since $u\in\{x,y,z\}$ then in particular for $u=x$ we have

$$\left(xy+xz+yz+x^2\right)(x-a)=0 \iff$$

$$(x-a)(x+z)(x+y)=0 \iff \left[\begin{array}{c} x=a \\ z=-x \\ y=-x \end{array}\right.$$

.

Consider case $x=a$.Then $\begin{cases} y+z=0 \\ \dfrac{1}{y}+\dfrac{1}{z}=0 \end{cases} \iff z=-y$ where $y\in\mathbb{R}\diagdown\{0\}$ is any. If $z=-x$ then $y=a$ and if $y=-x$ then $z=a$.So, we get solutions $(x,y,z)=(a,t,-t), t\in\mathbb{R}\diagdown\{0\}$.And by symmetry we have also $(x,y,z)=(t,a,-t),(t,-t,a),\ t\in\mathbb{R}\diagdown\{0\}$.

Solution 2.

$$\begin{cases} x+y+z=a \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{a} \end{cases} \iff \begin{cases} x+y=a-z \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a}-\dfrac{1}{z} \end{cases} \iff \begin{cases} x+y=a-z \\ \dfrac{z-a}{xy}+\dfrac{z-a}{az}=0 \end{cases} \iff$$

$$\begin{cases} x+y=a-z \\ \left[\begin{array}{c} z=a \\ xy+az=0 \end{array}\right. \end{cases} \iff \left[\begin{array}{l} \begin{cases} x+y=a-z \\ z=a \end{cases} \\ \begin{cases} x+y=a-z \\ xy=-az \end{cases} \end{array}\right. \iff \left[\begin{array}{l} \begin{cases} x+y=0 \\ z=a \end{cases} \\ \begin{cases} x=a \\ y=-z \end{cases} \\ \begin{cases} y=a \\ x=-z \end{cases} \end{array}\right. .$$

Problem 4.6 (95-Met. Rec.)
We have

$$\begin{cases} x+y+z=2 \\ xy+yz+zx=1 \end{cases} \iff \begin{cases} x+y=2-z \\ xy=1-z(y+z) \end{cases} \iff \begin{cases} x+y=2-z \\ xy=(z-1)^2 \end{cases}$$

Since $(x+y)^2-4xy=(x-y)^2$ then obtained Vieta's System have solutions iff $(2-z)^2-4(z-1)^2\geq 0 \iff z(3z-4)\leq 0 \iff z\in[0,4/3]$ and, due to symmetry, $x,y\in[0,4/3]$ as well.

Problem 4.7 (96-Met. Rec.)
Solution.
Note that

$$2(\cos x-\cos y)=\cos 2x\cos y \iff 2\cos x=\cos y(2+\cos 2x) \iff$$

$$2\cos x=\cos y\left(1+2\cos^2 x\right) \iff \cos y=\frac{2\cos x}{1+2\cos^2 x}.$$

Also note that $\dfrac{2|\cos x|}{1+2\cos^2 x}\leq\dfrac{1}{\sqrt{2}} \iff \left(\sqrt{2}|\cos x|-1\right)^2\geq 0 \iff |\cos x|,|\cos y|,|\cos z|\leq\dfrac{1}{\sqrt{2}}$.Since $\cos y=\dfrac{2\cos x}{1+2\cos^2 x} \iff \sqrt{2}\cos y=\dfrac{2\sqrt{2}\cos x}{1+\left(\sqrt{2}\cos x\right)^2}$ then, denoting, $u:=\sqrt{2}\cos x, v:=\sqrt{2}\cos y, w:=\sqrt{2}\cos z$ and $f(t):=\dfrac{2t}{1+t^2}$ can rewrite original system in the form

$$\textbf{(1)}\qquad \begin{cases} v=f(u)\\ w=f(v)\\ u=f(w)\end{cases},\quad \text{where } u,v,w\in[-1,1].$$

Note that $f(t)$ increasing in $[-1,1]$.Indeed, for $-1\leq t_1<t_2\leq 1$ we have

$$f(t_2)-f(t_1)=\frac{2t_2}{1+t_2^2}-\frac{2t_1}{1+t_1^2}=\frac{2(t_2-t_1)(1-t_1t_2)}{(t_2^2+1)(t_1^2+1)}>0$$

because $t_1t_2<1$.Since $f(t)$ increasing in $[-1,1]$ then u,v,w can be solution of **(1)** if $u=v=w$. Indeed, if we assume that $v\neq u$ then in case $u<v$ we obtain $f(u)<f(v) \iff v<w \implies f(v)<f(w) \iff w<u$ and, therefore, $u<v<w<u$,that is contradiction. If $u>v$ then $f(u)>f(v) \iff v>w \implies f(v)>f(w) \iff w>u$ and, therefore, $u>v>w>u$,that is contradiction again. So, $u=v=w=t$, where $t=0$ is only solution of equation $f(t)=0$.Thus, $\cos x=\cos y=\cos z=0 \iff x,y,z\in\{\pi/2+n\pi \mid n\in\mathbb{Z}\}$.

5. Functional equations and inequalities

Problem 5.1(97-Met. Rec.)
a) Note that $f\left(x^2\right)-(f(x))^2\geq 1/4 \iff$

$$f\left(x^2\right)-f(x)\geq 1/4-f(x)+(f(x))^2 \iff f\left(x^2\right)-f(x)\geq(f(x)-1/2)^2.$$

Let $x=0$ then

$$f\left(0^2\right)-f(0)\geq(f(0)-1/2)^2 \iff 0\geq(f(0)-1/2)^2 \iff f(0)=1/2;$$

Let $x=1$ then

$$f\left(1^2\right)-f(1)\geq(f(1)-1/2)^2 \iff 0\geq(f(1)-1/2)^2 \iff f(1)=1/2.$$

Thus, $f(1) = f(0)$ and that contradict to claim

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

So, there is no functions that satisfy to conditions of the problem.

b) Let $x = y = 0$. Then $f(0+0) \leq f(0) + f(0) \iff 0 \leq f(0)$. From the other hand since $f(x) \leq x$ for any $x \in \mathbb{R}$ we have $f(0) \leq 0$. Hence, $f(0) = 0$.By replacing y in inequality $f(x+y) \leq f(x) + f(y)$ with $-x$ we obtain

$$f(0) \leq f(x) + f(-x) \iff 0 \leq f(x) + f(-x) \implies$$

$$-f(x) \leq f(-x) \leq -x \implies -f(x) \leq -x \iff x \leq f(x)$$

and since $f(x) \leq x$ then $f(x) = x$.

Problem 5.2 (99-Met. Rec.)

Note that equation $f(x + f(x)) = f(x)$ have sence only if

$$x + f(x) \in [0,1] \iff -x \leq f(x) \leq 1 - x \implies -(x + f(x)) \leq f(x + f(x)) \leq 1 - (x + f(x)) \iff$$

$$-x - f(x) \leq f(x) \leq 1 - x - f(x) \iff -\frac{x}{2} \leq f(x) \leq \frac{1-x}{2} \iff$$

$$-\frac{x + f(x)}{2} \leq f(x + f(x)) \leq \frac{1 - (x + f(x))}{2} \iff -\frac{x + f(x)}{2} \leq f(x) \leq \frac{1 - x - f(x)}{2} \iff$$

$$-\frac{x}{3} \leq f(x) \leq \frac{1-x}{3}, \text{and so on....}$$

For any natural n, assuming $-\frac{x}{n} \leq f(x) \leq \frac{1-x}{n}$ for any $x \in [0,1]$, and by replacing x with $x + f(x) \in [0,1]$ we obtain

$$-\frac{x + f(x)}{n} \leq f(x + f(x)) \leq \frac{1 - (x + f(x))}{n} \iff$$

$$-\frac{x + f(x)}{n} \leq f(x) \leq \frac{1 - (x + f(x))}{n} \iff -\frac{x}{n+1} \leq f(x) \leq \frac{1-x}{n+1}.$$

Thus, by Math Induction, inequality $-\frac{x}{n} \leq f(x) \leq \frac{1-x}{n}, x \in [0,1]$ holds for any natural n.Hence, $\lim_{n\to\infty} \left(-\frac{x}{n}\right) \leq \lim_{n\to\infty} f(x) \leq \lim_{n\to\infty} \frac{1-x}{n} \iff 0 \leq f(x) \leq 0 \iff f(x) = 0$.

◊

Problem 5.3(100-Met. Rec.)

Note that

$$f(x)f(y)-xy=f(x)+f(y)-1 \iff f(x)f(y)-f(x)-f(y)+1=xy \iff (f(x)-1)(f(y)-1)=xy.$$

Let $y=1$. Then $(f(x)-1)(f(1)-1)=x$ for any $x\in\mathbb{R}$. In particular for $x=1$ we obtain

$$(f(1)-1)^2=1 \iff \left[\begin{array}{c} f(1)-1=1 \\ f(1)-1=-1 \end{array}\right. \iff \left[\begin{array}{c} f(1)=2 \\ f(1)=0 \end{array}\right. .$$

If $f(1)=2$ then $(f(x)-1)(f(1)-1)=x$ yields $f(x)-1=x \iff f(x)=x+1$

If $f(1)=2$ then $(f(x)-1)(f(1)-1)=x$ yields $(f(x)-1)(-1)=x \iff f(x)=1-x$.

Thus, functions $f(x)=x+1$ and $f(x)=1-x$ are all solutions of functional equation of the problem.

Remark.

Continuity requirement in the problem is unnecessary.

Problem5.4 (101-Met. Rec.)

By replacing in equation x with $\dfrac{x}{n}$ we obtain that

$$nf(x)=f\left(\frac{x}{n}\right)+x \iff f(x)=\frac{1}{n}f\left(\frac{x}{n}\right)+\frac{x}{n}$$

for any real x. Then

$$f\left(\frac{x}{n}\right)=\frac{1}{n}f\left(\frac{x}{n^2}\right)+\frac{x}{n^2} \iff \frac{1}{n}f\left(\frac{x}{n}\right)=\frac{1}{n^2}f\left(\frac{x}{n^2}\right)+\frac{x}{n^3}.$$

Hence,

$$f(x)+\frac{1}{n}f\left(\frac{x}{n}\right)=\frac{1}{n}f\left(\frac{x}{n}\right)+\frac{x}{n}+\frac{1}{n^2}f\left(\frac{x}{n^2}\right)+\frac{x}{n^3} \iff f(x)=\frac{1}{n^2}f\left(\frac{x}{n^2}\right)+\frac{x}{n}+\frac{x}{n^3}.$$

And again by replacing x in $f(x)=\dfrac{1}{n}f\left(\dfrac{x}{n}\right)+\dfrac{x}{n}$ with $\dfrac{x}{n^2}$ we obtain

$$f\left(\frac{x}{n^2}\right)=\frac{1}{n}f\left(\frac{x}{n^3}\right)+\frac{x}{n^3}.$$

Hence,

$$f(x)+\frac{1}{n^2}f\left(\frac{x}{n^2}\right)=\frac{1}{n^2}f\left(\frac{x}{n^2}\right)+\frac{x}{n}+\frac{x}{n^3}+\left(\frac{1}{n^3}f\left(\frac{x}{n^3}\right)+\frac{x}{n^5}\right) \iff$$

$$f(x)=\frac{1}{n^3}f\left(\frac{x}{n^3}\right)+\frac{x}{n}+\frac{x}{n^3}+\frac{x}{n^5} \text{ and so on....}$$

For any $k \in \mathbb{N}$ assuming that $f(x) = \dfrac{1}{n^k} f\left(\dfrac{x}{n^k}\right) + \sum_{i=1}^{k} \dfrac{x}{n^{2i-1}}$.

Then by replacing x in $f(x) = \dfrac{1}{n} f\left(\dfrac{x}{n}\right) + \dfrac{x}{n}$ with $\dfrac{x}{n^k}$ we obtain

$$f\left(\frac{x}{n^k}\right) = \frac{1}{n} f\left(\frac{x}{n^{k+1}}\right) + \frac{x}{n^{k+1}} \iff \frac{1}{n^k} f\left(\frac{x}{n^k}\right) = \frac{1}{n^{k+1}} f\left(\frac{x}{n^{k+1}}\right) + \frac{x}{n^{2k+1}}$$

and, therefore,

$$f(x) + \frac{1}{n^k} f\left(\frac{x}{n^k}\right) = \frac{1}{n^k} f\left(\frac{x}{n^k}\right) + \sum_{i=1}^{k} \frac{x}{n^{2i-1}} + \frac{1}{n^{k+1}} f\left(\frac{x}{n^{k+1}}\right) + \frac{x}{n^{2k+1}} \iff$$

$$f(x) = \frac{1}{n^{k+1}} f\left(\frac{x}{n^{k+1}}\right) + \sum_{i=1}^{k+1} \frac{x}{n^{2i-1}}.$$

Thus, by Math Induction $f(x) = \dfrac{1}{n^k} f\left(\dfrac{x}{n^k}\right) + \sum_{i=1}^{k} \dfrac{x}{n^{2i-1}}$ for any $k \in \mathbb{N}$.Then,

since $\lim_{k\to\infty} \sum_{i=1}^{k} \dfrac{x}{n^{2i-1}} = x \lim_{k\to\infty} \dfrac{\frac{1}{n} - \frac{1}{n^{2k+1}}}{1 - \frac{1}{n^2}} = \dfrac{nx}{n^2-1}$ and $\lim_{k\to\infty} \dfrac{1}{n^{k+1}} f\left(\dfrac{x}{n^{k+1}}\right) =$ 0 ($\lim_{k\to\infty} f\left(\dfrac{x}{n^{k+1}}\right) = f\left(\lim_{k\to\infty} \dfrac{x}{n^{k+1}}\right) = f(0)$ (due continuity of f in $x = 0$) and $\lim_{k\to\infty} \dfrac{1}{n^{k+1}} = 0$) we obtain that $f(x) = \dfrac{nx}{n^2-1}$.

Remark.

And again *Continuity requirement in the problem is unnecessary. Suffice claim that f is boundeed in some neighborhood of* 0.

Problem 5.5(14-Met. Rec.)

Suppose that there is function continuous on $\mathbb{R}$ such that $f(x+1)(f(x)+1)+1 = 0$. First not that $f(x+1)(f(x)+1)+1 = 0$ yields $f(x) \neq -1$ for any $x \in \mathbb{R}$ because otherwise if $f(x_0) = 0$ for some x_0 then we get $f(x_0+1)(-1+1)+1 = 0 \iff 1 = 0$. Furtermore, $f(x) \neq 0$ for any $x \in \mathbb{R}$,because otherwise if $f(x_0) = -1$ for some x_0 then we obtain $f(x_0+1)(0+1)+1 = 0 \iff f(x_0+1) = -1$ and that contradict to previous conclusion. Hence, since $f(x)$ is continuous on $\mathbb{R}$ then $f(x)$ preserve sign on $\mathbb{R}$.

From the other hand since

$$f(x+1) = -\frac{1}{f(x)+1}, x \in \mathbb{R} \text{ then } f(x+2) = -\frac{1}{f(x+1)+1} = -\frac{1}{-\frac{1}{f(x)+1}+1} = -1-\frac{1}{f(x)}$$

.If $f(x) > 0$ for all $x \in \mathbb{R}$ then since $f(x+2) = -1 - \dfrac{1}{f(x)} < 0$ we get contradiction. If $f(x) < 0$ for all $x \in \mathbb{R}$ then in the case $-1 < f(x) < 0$

we have $f(x+2) = -1 - \dfrac{1}{f(x)} > 0$, that is contradiction again; In the case $f(x) < -1$ we obtain $f(x+1) = -\dfrac{1}{f(x)+1} > 0-$contradiction.

Problem 5.6(15-Met.Rec.)

For given $n \in \mathbb{N}$ any natural a can be uniquely represented in the form*

$$a = k(n+1) + r, \text{ where } k \in \mathbb{N} \cup \{0\}, r \in \{1, 2, ..., n+1\}$$

.

Let $a \in \mathbb{N}$ be any. If $a \geq n+2$ then $k \geq 1$ and $f(a) = f(k(n+1)+r) = f(k(n+1)+r-1+1)$. Applying $f(m+k) = f(mk-n)$ for $m = k(n+1) + r - 1, k = 1$ we obtain

$$f(a) = f((k(n+1)+r-1)\cdot 1 - n) = f(k(n+1)+r-1-n) =$$

$$f(k(n+1)+r-1-n) = f((k-1)(n+1)+r).$$

Thus, if $a \geq n+2 \iff k \geq 1$ then

$$f(a) = f(a-(n+1)) = ... = f(a-k(n+1)) = f(r).$$

Let $a \in \{1, 2, ..., n+1\}$ then $k = 0$ and $a = r$.We will prove that $f(r) = f(1)$ for any $r \in \{1, 2, ..., n+1\}$.Since $r \geq 1$ then by replacing $k \in \mathbb{N}$ in $f(k(n+1)+r) = f(r)$ with r we obtain $f(r(n+1)+r) = f(r)$ and applying $f(m+k) = f(mk-n)$ for $m = r(n+1), k = r$ we get

$$f(r) = f(r(n+1)+r) = f\left(r^2(n+1)-n\right) = f\left(r^2(n+1)-n-1+1\right) =$$

$$f\left(\left(r^2-1\right)(n+1)+1\right) = f(1).$$

*__Remark.__

If $a, b \in \mathbb{N}$ then there is unique pair (k, r) of integers such that $a = kb + r$ and $1 \leq r \leq b$.Indeed, by Representation Theorem (division with remainder) there are unique integer number p, ρ such that $a = pb + \rho$ and $0 \leq \rho < b$. If $\rho \neq 0$ then $r := \rho$ and $k := p$; If $\rho = 0$ then $a = pb \iff a = (p-1)b + b$ and $k := p-1, r := b$.

We will prove uniqueness.

Let $\begin{cases} a = kb + r \\ 1 \leq r \leq b \end{cases}$ and $\begin{cases} a = k_1 b + r_1 \\ 1 \leq r_1 \leq b \end{cases}$ then $kb + r = k_1 b + r_1 \implies b|k-k_1| = |r-r_1|$.Since $1 - b \leq r - r_1 \leq b - 1$ then $|r-r_1| < b$.Hence, $b|k-k_1| < b \iff |k-k_1| < 1 \iff k = k_1 \implies r = r_1$.

★Problem 5.7(U182)

Let $x \in \left(\dfrac{1}{2}, 1\right)$.Define sequence $(x_n)_{n\geq 0}$ as follows:

$x_0 := x$ and $x_n = 2x_{n-1} - 1, n \geq 1.$

Then for any $x \in \left(\dfrac{1}{2}, 1\right)$ there is n such that $x_n \in \left[0, \dfrac{1}{2}\right]$. Indeed, from $x_{n+1} = 2x_n - 1 \iff$

$x_{n+1} - 1 = 2(x_n - 1)$ follows $x_n - 1 = 2^n (x_0 - 1) \iff x_n = 1 - 2^n (1 - x)$.

Since $0 \leq x_n \leq \dfrac{1}{2} \iff 0 \leq 1 - 2^n (1 - x) \leq \dfrac{1}{2} \iff \begin{cases} 2^n \leq \dfrac{1}{1-x} \\ \dfrac{1}{1-x} \leq 2^{n+1} \end{cases} \iff$

$$\log_2 \frac{1}{1-x} - 1 \leq n \leq \log_2 \frac{1}{1-x} \text{ then } \left\lfloor \log_2 \frac{1}{1-x} \right\rfloor$$

is such n, (because for any real a by definition of integer part of a we have $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1 \iff a - 1 < \lfloor a \rfloor \leq a$). Hence, for such n we obtain $f(x) = f(x_0) = f(2x_0 - 1) = f(x_1) = f(2x_1 - 1) = f(x_2) = ... = f(x_{n-1}) = f(2x_{n-1} - 1) = f(x_n) = c.$

Thus $f(x) = c$ for any $x \in [0, 1)$ and, since by condition $f(x)$ is continuous on $[0, 1]$, then $f(1) = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} c = c.$

6. Recurrences.

Problem 6.1(4-Met. Rec.)

Let (x, y) be pair of two coprime natural numbers such that $\dfrac{x^2 + p}{y}$and $\dfrac{y^2 + p}{x}$ are integer numbers and let $t := \dfrac{x^2 + p}{y} \in \mathbb{N}$. Such pairs we will call "exotic". Assume also that $x > y$ and $\gcd(x, p) = \gcd(y, p) = 1$ We will prove that $t > x$ and (t, x) be exotic pair, that is $\gcd(t, x) = 1, \gcd(t, p) = 1$ and $\dfrac{t^2 + p}{x}$, $\dfrac{x^2 + p}{t}$ are integer numbers.

1. $t > x \iff \dfrac{x^2 + p}{y} > x \iff x^2 + p > xy \iff x(x - y) + p > 0$ (since $x > y$);

2. Since $\gcd(y, p) = 1$ and $\gcd\left(x^2 + p, p\right) = \gcd\left(x^2, p\right) = 1$ then

$\gcd(t, p) = \gcd\left(\dfrac{x^2 + p}{y}, p\right) = \gcd\left(y \cdot \dfrac{x^2 + p}{y}, p\right) = \gcd\left(x^2 + p, p\right) = 1.$

(Here we used two folowing properties of gcd :

i. Preservation Lemma: $\gcd(a, b) = \gcd(a - kb, b)$ for any integer k;

ii. Cancellation Property: If $\gcd(c, b) = 1$ then $\gcd(a, b) = \gcd(ac, b)$.)

3. $\dfrac{t^2 + p}{x} \in \mathbb{N}$. Indeed, $\dfrac{t^2 + p}{x} = \dfrac{\left(\dfrac{x^2 + p}{y}\right)^2 + p}{x} = \dfrac{p^2 + 2px^2 + py^2 + x^4}{xy^2}$

and $\dfrac{p^2+2px^2+py^2+x^4}{x}=2px+x^3+p\cdot\dfrac{y^2+p}{x}\in\mathbb{N}$, $\dfrac{p^2+2px^2+py^2+x^4}{y^2}=$ $p+\left(\dfrac{x^2+p}{y}\right)^2\in\mathbb{N}$ then $p^2+2px^2+py^2+x^4\,\vdots\,xy^2$ because $\gcd\left(x,y^2\right)=1$.

(If $a\,\vdots\,b,\ a\,\vdots\,c$ and $\gcd(b,c)=1$ then $a\,\vdots\,bc$).

Also, obvious that $\dfrac{x^2+p}{t}=y\in\mathbb{N}$.

Thus from exotic pair (x,y) we obtain new exotic pair (t,x) with $t>x$ and so on.... This process is infinite. To complete solution we have to prove existance of exotic pair. Easy to check that pair $(x,y):=(p+1,1)$ is exotic. Indeed, $\dfrac{x^2+p}{y}\in\mathbb{N}$ because $y=1$, $\dfrac{y^2+p}{x}=1\in\mathbb{N}$ and $\gcd(p+1,1)=\gcd(p+1,p)=\gcd(1,p)$ .Thus, if $x_0:=1, x_1:=p+1$ and $x_{n+1}=\dfrac{x_n^2+p}{x_{n-1}}, n\in\mathbb{N}$ then we get increasing sequence $x_0,x_1,...,x_n,...$such that any pair (x_{n+1},x_n) is exotic. Since $x_{n+1}=\dfrac{x_n^2+p}{x_{n-1}}\iff x_n^2-x_{n+1}x_{n-1}+p=0, n\in\mathbb{N}$ then

$$x_{n+1}^2-x_{n+2}x_n=x_n^2-x_{n+1}x_{n-1}\iff x_{n+1}^2+x_{n+1}x_{n-1}=x_n^2+x_{n+2}x_n\iff$$

$$x_{n+1}\left(x_{n+1}+x_{n-1}\right)=x_n\left(x_{n+2}+x_n\right)\iff\frac{x_{n+1}+x_{n-1}}{x_n}=\frac{x_{n+2}+x_n}{x_{n+1}}, n\in\mathbb{N}\implies$$

$$\frac{x_{n+1}+x_{n-1}}{x_n}=\frac{x_2+x_0}{x_1}=\frac{p^2+3p+1+1}{p+1}=\frac{(p+1)(p+2)}{p+1}=p+2.$$

Thus the sequence $x_0,x_1,...,x_n,...$ in reality defined by Linear Homogeneous Recurrence

$$\begin{cases} x_{n+1}-(p+2)\ x_n+x_{n-1}=0, n\in\mathbb{N} \\ x_0=1, x_1=p+1\end{cases}$$

of the second degree with constant coefficients.

By the way appears the following problem:

Find sufficient and necessity conditions for x_0 and x_1 for which all terms of the sequence

$x_0,x_1,...,x_n,$..defined by recurrence $x_{n+1}=\dfrac{x_n^2+p}{x_{n-1}}, n\in\mathbb{N}$ be integer numbers.

Problem 6.2(5-Met.Rec.)

First of all we note that $a_n\neq 0$ for any $n\in\mathbb{N}$. It can be easy proved by Math Induction.

1. Base of Math Induction: $a_1, a_2, a_3 > 0$;
2. Step of Math Induction: For any $n \in \mathbb{N}$ assuming $a_n,\ a_{n+1}, a_{n+2} > 0$ we obtain $a_{n+3} = \dfrac{a_{n+1}a_{n+2}+5}{a_n} > 0$.Thus, $a_n,\ a_{n+1}, a_{n+2} > 0 \implies a_{n+1}, a_{n+2}, a_{n+3} > 0$.Since $a_{n+3} = \dfrac{a_{n+1}a_{n+2}+5}{a_n} \iff a_{n+3}a_n - a_{n+1}a_{n+2} = 5, n \in \mathbb{N}$ then for any $n \geq 2$ we have

$$a_{n+3}a_n - a_{n+1}a_{n+2} = a_{n+2}a_{n-1} - a_n a_{n+1} \iff (a_{n+3}+a_{n+1})\,a_n = a_{n+2}\,(a_{n+1}+a_{n-1}) \iff$$

$$\frac{a_{n+3}+a_{n+1}}{a_{n+2}} = \frac{a_{n+1}+a_{n-1}}{a_n}.$$

That yields $\dfrac{a_{n+1}+a_{n-1}}{a_n} = \begin{cases} \dfrac{a_3+a_1}{a_2} & \text{if } n \text{ is even} \\ \dfrac{a_4+a_2}{a_3} & \text{if } n \text{ is odd} \end{cases}$.

Noting that $a_4 = \dfrac{2\cdot 1+5}{1} = 7$ we obtain

$$\frac{a_{n+1}+a_{n-1}}{a_n} = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases} = \frac{1}{2}\left(7+(-1)^{n-1}\right), n \geq 2.$$

Thus, sequence (a_n) can be defined as follows

$$\begin{cases} a_1 = a_2 = 1, a_3 = 2 \\ a_{n+1} = \dfrac{1}{2}\left(7+(-1)^{n-1}\right)a_n - a_{n-1}\ , n \geq 2 \end{cases}$$

and, therefore, $a_n \in \mathbb{Z}$ for any all $n \in \mathbb{N}$.

Remark.

In connection with **Problem 4** (final stage) and **Problem 5** we can consider the following

★ **Problem.**

Prove that for any natural number p there are infinitely many triples (x, y, z) of distinct natural
numbers such that:

i. $\gcd(x,p) = \gcd(y,p) = \gcd(z,p) = 1$;

ii. $\dfrac{xy+p}{z}$ and $\dfrac{yz+p}{x}$ are integer.

Solution.

Assume that we allready have a triple (x, y, z) of distinct natural numbers that satisfy to **i.** and **ii.** And in addition assume that $x > y > z$, $\dfrac{x+z}{y}$ is integer and x, y, z are pairly coprime, that is $\gcd(x,y) = \gcd(y,z) = \gcd(z,x) = 1$. Let $t := \dfrac{xy+p}{z}$ and we will show that triple (t, x, y) has the same properties as the triple (x, y, z) ,that is:

1. $t > x > y$ (suffice to prove $t > x$);

2. $\gcd(t,p) = \gcd(x,p) = \gcd(y,p) = 1$(suffice to prove $\gcd(t,p) = 1$);

3. t, x, y are pairly coprime;
4. $\dfrac{t+y}{x}, \dfrac{tx+p}{y}, \dfrac{xy+p}{t}$ are integer.
Indeed:
1. $\dfrac{xy+p}{z} > x \iff xy+p > xz \iff x(y-z)+p > 0$ (since $y-z>0$);
2. $\gcd(t,p) = \gcd\left(\dfrac{xy+p}{z}, p\right) = \gcd\left(z\cdot\dfrac{xy+p}{z}, p\right) = \gcd(xy+p,p) =$
$\gcd(xy,p) = 1$because $\gcd(x,p) = \gcd(y,p) = \gcd(z,p) = 1$;
3. Suffice to prove $\gcd(t,x) = \gcd(t,y) = 1$ because $\gcd(x,y) = 1$.
We have $\gcd(t,x) = \gcd\left(\dfrac{xy+p}{z}, x\right) = \gcd\left(z\cdot\dfrac{xy+p}{z}, x\right) =$
$\gcd(xy+p,x) = \gcd(x,p) = 1$ and, similarly, $\gcd(t,y) = 1$.
4. Since $\dfrac{t+y}{x} = \dfrac{\dfrac{xy+p}{z}+y}{x} = \dfrac{xy+p+yz}{xz}$ and $\dfrac{xy+p+yz}{x} =$
$y + \dfrac{yz+p}{x} \in \mathbb{Z}, \dfrac{xy+p+yz}{z} = y + \dfrac{xy+p}{z} \in \mathbb{Z}$ then $xy+p+yz \;\vdots\; xz$
(because $\gcd(z,x) = 1$), $\dfrac{tx+p}{y} = \dfrac{\dfrac{xy+p}{z}\cdot x+p}{y} = \dfrac{yx^2+px+pz}{yz}$
and $\dfrac{yx^2+px+pz}{y} = x^2 + p\cdot\dfrac{x+z}{y} \in \mathbb{Z}$ (since$\dfrac{x+z}{y} \in \mathbb{Z}$),
$\dfrac{yx^2+px+pz}{z} = p + x\cdot\dfrac{yx+p}{z} \in \mathbb{Z}$ (since $\dfrac{yx+p}{z} \in \mathbb{Z}$).
Hence, $yx^2+px+pz \;\vdots\; yz \implies \dfrac{tx+p}{y} \in \mathbb{Z}$.And at last $\dfrac{xy+p}{t} = z \in \mathbb{Z}$.

So, starting with triple (x,y,z) we construct new triple (t,x,y) which has the same properties as (x,y,z) and since $z<y<x<t$ the process of construction of triples can be continued indefinitely. Thus, everything reduced to finding at least one triple that satisfy to our claims.

Easy to see that $(x,y,z) = (2p+1, p+1, 1)$ satisfy to these claims and then, sequence (a_n) defined by recurrence $a_{n+3} = \dfrac{a_{n+1}a_{n+2}+p}{a_n}, n \in \mathbb{N}$ with initial conditions $a_1, a_2 = p+1, a_3 = 2p+1$ provide us infinitely many triples (a_{n+2}, a_{n+1}, a_n) satisfying the problem.

Problem 6.3(16-Met.Rec., Problem 5, Czechoslovakia, MO 1986)

Let $\overline{a}_n$ be some solution of the recurrence $a_{n+2} - 2a_{n+1} + a_n = 2, n \in \mathbb{N}$. For example $\overline{a}_n := n^2$ satisfy to recurrence $\left((n+2)^2 - 2(n+1)^2 + n^2 = 2\right)$. Then $b_n := a_n - n^2$ satisfy to recurrence $b_{n+2} - 2b_{n+1} + b_n = 0, n \in \mathbb{N}$ which has general solution $b_n = dn + c$.(Indeed, since $b_{n+2} - 2b_{n+1} + b_n = 0 \iff b_{n+2} - b_{n+1} = b_{n+1} - b_n, n \in \mathbb{N}$ then $b_{n+1} - b_n = b_2 - b_1, n \in \mathbb{N}$.That is (b_n) is arithmatic sequence with common difference $d := b_2 - b_1$ and therefore $b_n = dn + c$ for some ct).Hence, $a_n = n^2 + dn + c, n \in \mathbb{N}$ and we have $a_1 = 1 \iff 1^2 + d\cdot 1 + c = 1 \iff d = -c$. Hence, $a_n = n^2 - c(n-1)$. Since by condition of the problem a_n is integer for all $n \in \mathbb{N}$ then $c \in \mathbb{Z}$. (Indeed, $a_2 = 4 - c \implies c \in \mathbb{Z}$).

Now we can complete the solution.

Equation $a_n a_{n+1} = a_m$ we can consider as quadratic equation with respect to m in natural numbers. We have

$$a_n a_{n+1} = a_m \iff \left(n^2 - c(n-1)\right)\left((n+1)^2 - cn\right) = m^2 - c(m-1) \iff$$

$$m^2 - cm + c - \left(n^2 - c(n-1)\right)\left((n+1)^2 - cn\right) = 0 \iff$$

$$m^2 - cm + 2(c-1)n^3 - n^4 - \left(c^2 - c + 1\right)n^2 + c(c-1)n = 0 \iff$$

$$m^2 - cm + \left(c - cn + n + n^2\right)\left(cn - n^2 - n\right) = 0 \iff \left[\begin{array}{c} m = n^2 + n - c(n-1) \\ m = n(c - (n+1)) \end{array}\right. .$$

If $c \geq n+2$ then $n(c-(n+1)) > 0$ and we can take $m = cn - \left(n^2 + n\right) \in \mathbb{N}$;
If $c \leq n+1$ then $n^2 + n - c(n-1) \geq n^2 + n - (n+1)(n-1) = n+1$ and we can take $m = n^2 + n - c(n-1)$.

Problem 6.4(17 Met.Rec.)

First note that $a_{n+1} > a_n + 1, n \in \mathbb{N}$.Indeed, $a_2 - a_1 > 1$ and for any $n \in \mathbb{N}$ assuming $a_{k+1} > a_k + 1, k = 1, 2, ..., n$ we obtain $a_{n+2} - a_{n+1} = a_{n+1}^2 - a_{n+1} - a_n > a_{n+1}^2 - 2a_{n+1} + 1 = (a_{n+1} - 1)^2 > a_n^2 > 1$.For any term a_n of the sequence $(a_n)_{n\geq 1}$ we set in correspondence remainder from division a_n by 1986, that is $a_n \mapsto r_{1986}(a_n), n \in \mathbb{N}$. Further we will use short notation $r_n := r_{1986}(a_n)$. Then to each pair (a_n, a_{n+1}) we set in correspondence pair of its remainders $(r_n, r_{n+1}), n \in \mathbb{N}$. Since set of all pairs (a_n, a_{n+1}) is infinite (because $(a_n)_{n\geq 1}$ is strictly increasing) and set of pairs (r_n, r_{n+1}) is finite (because $r_n \in \{0, 1, ..., 1986\}$ for any n) then there are at least two natural k, m such that $(r_k, r_{k+1}) = (r_m, r_{m+1})$.Assume that $k < m$ and let $p := m - k$.

Note that sequence $r_1, r_2, ..., r_n, ...$defined recursively as follows:

$$\left\{\begin{array}{c} r_1 = 39, r_2 = 45 \\ r_{n+2} = r_{1986}\left(r_{n+1}^2 - r_n\right), n \in \mathbb{N} \end{array}\right.$$

.

From the other hand note that $r_{i-1} = r_{1986}\left(r_i^2 - r_{i+1}\right), i > 1$.Indeed,

$$r_i^2 - r_{i+1} \equiv a_i^2 - a_{i+1} \pmod{1986} \equiv a_{i-1} \pmod{1986} \equiv r_{i-1} \pmod{1986}$$

and, since $0 \leq r_{1986}\left(r_i^2 - r_{i+1}\right) < 1986$ we obtain $r_{i-1} = r_{1986}\left(r_i^2 - r_{i+1}\right)$.

If $k > 1$ then applying this "back (reverse) recurcion" to (r_k, r_{k+1}) and (r_m, r_{m+1}) we obtain $(r_{k-1}, r_k) = (r_{m-1}, r_m)$.Repeating this procedure $k - 1$ times we obtain

$$(r_1, r_2) = (r_{m-k+1}, r_{m-k+2}) \iff (r_1, r_2) = (r_{p+1}, r_{p+2}).$$

Then using recurrence $r_{n+2} = r_{1986}\left(r_{n+1}^2 - r_n\right), n \in \mathbb{N}$ we obtain that $(r_1, r_2, r_3, ..., r_p) = (r_{p+1}, r_{p+2}, r_{p+3}, ..., r_{2p})$ and futhermore, by Math Induction we can prove that $(r_{pi+1}, r_{pi+2}) = (r_1, r_2)$ for any $i \in \mathbb{N}$. Having $(r_{p+1}, r_{p+2}) = (r_1, r_2)$ as Base of Math Induction and in supposition $(r_{pi+1}, r_{pi+2}) = (r_1, r_2)$ and using $r_{n+2} = r_{1986}\left(r_{n+1}^2 - r_n\right), n \in \mathbb{N}$ we obtain

$$\left(r_{pi+1}, r_{pi+2}, r_{pi+3}, ..., r_{p(i+1)}, r_{p(i+1)+1}, r_{p(i+1)+2}\right) = (r_1, r_2, r_3, ..., r_p, r_{p+1}, r_{p+2})$$

and, therefore, $\left(r_{p(i+1)+1}, r_{p(i+1)+2}\right) = (r_{p+1}, r_{p+2}) = (r_1, r_2)$.

Thus, sequence $r_1, r_2, r_3, ..., r_n, ...$ is periodic with period p,that is for any $n, i \in \mathbb{N}$ holds $r_{n+pi} = r_n$.In particular, since $r_3 = r_{1986}(a_3) = r_{1986}\left(45^2 - 39\right) = r_{1986}(1986) = 0$ then $r_{3+pi} = r_3 = 0$ for any $i \in \mathbb{N} \iff a_{3+pi} \vdots 1986$ for any $i \in \mathbb{N}$.

$\Diamond$

Problem 6.5(31-Met. Rec.)

So, we have $a_0 = a, b_0 = b, c_0 = c, d_0 = d$ and

$$\textbf{(1)} \qquad \begin{cases} a_{n+1} = a_n - b_n \\ b_{n+1} = b_n - c_n \\ c_{n+1} = c_n - d_n \\ d_{n+1} = d_n - a_n \end{cases}, n \in \mathbb{N} \cup \{0\}.$$

Obvious that for any $n \in \mathbb{N}$ holds $a_n + b_n + c_n + d_n = 0$.

Also we have $a_{n+1} + c_{n+1} = a_n - b_n + c_n - d_n \iff a_{n+1} + c_{n+1} = a_n + c_n - (b_n + d_n)$ and since $-(b_n + d_n) = a_n + c_n$ then $a_{n+1} + c_{n+1} = 2(a_n + c_n), n \in \mathbb{N} \iff a_n + c_n = 2^{n-1}(a_1 + c_1), n \in \mathbb{N}$. From the other hand since $a_{n+1} = a_n - b_n \iff b_n = a_n - a_{n+1} \implies b_{n+1} = a_{n+1} - a_{n+2}$ then $c_n = b_n - b_{n+1}$ becomes $c_n = a_n - a_{n+1} - (a_{n+1} - a_{n+2}) \iff c_n = a_n - 2a_{n+1} + a_{n+2}$.

Hence, $a_n + c_n = 2^{n-1}(a_1 + c_1) \implies a_n + a_n - 2a_{n+1} + a_{n+2} = 2^{n-1}(a_1 + c_1) \iff$

(2) $\quad a_{n+2} - 2a_{n+1} + 2a_n = 2^{n-1}p, n \in \mathbb{N}$, where $p = a_1 + c_1 = a - b + c - d$

Since $2^{n-1} = 2^n - 2 \cdot 2^{n-1} + 2 \cdot 2^{n-2}$ then

(2) $\iff a_{n+2} - 2^n p - 2\left(a_{n+1} - 2^{n-1}p\right) + 2\left(a_n - 2^{n-1}p\right) = 0 \iff$

$$\frac{a_{n+2} - 2^n p}{\left(\sqrt{2}\right)^{n+2}} - 2 \cdot \frac{1}{\sqrt{2}} \frac{a_{n+1} - 2^{n-1}p}{\left(\sqrt{2}\right)^{n+1}} + 2 \cdot \frac{a_n - 2^{n-1}p}{2 \cdot \left(\sqrt{2}\right)^n} = 0 \iff$$

$$x_{n+2} - 2\cos\frac{\pi}{4} \cdot x_{n+1} + x_n = 0, \text{ where } x_n := \frac{a_n - 2^{n-1}p}{\left(\sqrt{2}\right)^n}.$$

Since general solution of the homogeneous recurrence

$$\textbf{(3)} \qquad x_{n+2} - 2\cos\frac{\pi}{4} \cdot x_{n+1} + x_n = 0, n \in \mathbb{N} \cup \{0\}$$

is $x_n = \alpha\cos\frac{n\pi}{4} + \beta\sin\frac{n\pi}{4}$, $n \in \mathbb{N} \cup \{0\}$ then $a_n - 2^{n-1}p = \left(\sqrt{2}\right)^n\left(\alpha\cos\frac{n\pi}{4} + \beta\sin\frac{n\pi}{4}\right) \iff a_n = \left(\sqrt{2}\right)^n\left(\alpha\cos\frac{n\pi}{4} + \beta\sin\frac{n\pi}{4}\right) + 2^{n-2}p, n \in \mathbb{N} \cup \{0\}$.

Since $a_1 = \left(\sqrt{2}\right)^1 \left(\alpha \cos \dfrac{1 \cdot \pi}{4} + \beta \sin \dfrac{1 \cdot \pi}{4}\right) + 2^{-1} p \iff$

$$a_1 = \sqrt{2}\left(\alpha \cdot \frac{1}{\sqrt{2}} + \beta \cdot \frac{1}{\sqrt{2}}\right) + \frac{a_1 + c_1}{2} \iff a_1 = \alpha + \beta + \frac{a_1 + c_1}{2} \iff \alpha + \beta = \frac{a_1 - c_1}{2}$$

and

$$a_2 = \left(\sqrt{2}\right)^2 \left(\alpha \cos \frac{2 \cdot \pi}{4} + \beta \sin \frac{2 \cdot \pi}{4}\right) + 2^0 p \iff$$

$$a_1 - b_1 = 2\beta + (a_1 + c_1) \iff -b_1 - c_1 = 2\beta \iff$$

$$\beta = -\frac{b_1 + c_1}{2} = -\frac{b - c + c - d}{2} = \frac{d - b}{2}$$

then

$$\alpha = -\beta - c_1 = \frac{b_1 + c_1}{2} + \frac{a_1 - c_1}{2} = \frac{a_1 + b_1}{2} = \frac{a - b + b - c}{2} = \frac{a - c}{2}.$$

Hence,

$$a_n = \left(\sqrt{2}\right)^n \left(\frac{a - c}{2} \cos \frac{n\pi}{4} + \frac{d - b}{2} \sin \frac{n\pi}{4}\right) + 2^{n-2}(a + c - b - d) =$$

$$2^{\frac{n-2}{2}} \left((a - c) \cos \frac{n\pi}{4} + (d - b) \sin \frac{n\pi}{4}\right) + 2^{n-2}(a + c - b - d).$$

By cyclic symmetry we also have

$$c_n = 2^{\frac{n-2}{2}} \left((c - a) \cos \frac{n\pi}{4} + (b - d) \sin \frac{n\pi}{4}\right) + 2^{n-2}(c + a - d - b)$$

and applying formula $\max\{x, y\} = \dfrac{1}{2}(x + y + |x - y|)$ we obtain

$$\max\{a_n, c_n\} = 2^{n-2}(a + c - b - d) + 2^{\frac{n-2}{2}} \left|(a - c) \cos \frac{n\pi}{4} + (d - b) \sin \frac{n\pi}{4}\right|$$

and cyclic

$$\max\{b_n, d_n\} = 2^{n-2}(b + d - c - a) + 2^{\frac{n-2}{2}} \left|(b - d) \cos \frac{n\pi}{4} + (a - c) \sin \frac{n\pi}{4}\right|$$

To solve the problem suffice to prove that $\mu_n := \max\{a_n, b_n, c_n, d_n\} \geq 2^{\frac{n-2}{2}}$ for any, divisible by four, natural n, that is prove the inequality

(4) $\quad \mu_{4k} \geq 2^{2k-1}, k \in \mathbb{N}$.

Since for $n = 4k$ we have $\cos \dfrac{n\pi}{4} = \cos k\pi = (-1)^k$ and $\sin \dfrac{n\pi}{4} = \sin k\pi = 0$ then $\mu_{4k} = 2^{2k-1} \max\left\{2^{2k-1}(a + c - b - d) + |a - c|, 2^{2k-1}(b + d - a - c) + |b - d|\right\}$ and $\mu_{4k} \geq 2^{2k-1}, k \in \mathbb{N} \iff$

(5) $\max\left\{2^{2k-1}(a+c-b-d)+|a-c|,2^{2k-1}(b+d-a-c)+|b-d|\right\}\geq$ 1.

If $a+c\neq b+d$ then $\max\left\{2^{2k-1}(a+c-b-d)+|a-c|,2^{2k-1}(b+d-a-c)+|b-d|\right\}\geq$ $\max\left\{2^{2k-1}(a+c-b-d),2^{2k-1}(b+d-a-c)\right\}+\min\{|a-c|,|b-d|\}=$ $2^{2k-1}|a+c-b-d|+\min\{|a-c|,|b-d|\}\geq 2^{2k-1}|a+c-b-d|\geq$ $2^{2k-1}\geq 2>1$.

If $a+c=b+d$ then inequality **(5)** becomes $\max\{|a-c|,|b-d|\}\geq 1$.

If $a\neq c$ or $b\neq d$ inequality **(5)** obviously holds. The case when $a=c$ and $b=d$ impossible because then since $a+c=b+d$ we obtain $a=b=c=d$. Thus, inequalty **(5)** proved and since $\mu_{4k}\geq 2^{2k-1},k\in\mathbb{N}$ then in particular we have $\mu_{100}\geq 2^{2\cdot 25-1}=2^{49}>10^9$ ($2^{49}>2^{30}=\left(2^{10}\right)^3>1000^3=10^9$).

Analysis and generalization.

We will prove more general statement, namely we will prove that

$\max\{a_n,b_n,c_n,d_n\}\geq 2^{\frac{n-2}{2}}$ for any natural $n\geq 2$.

Proof.

Since $a_n+c_n=-b_n-d_n,n\in\mathbb{N}$ then $a_{n+1}+c_{n+1}=a_n-b_n+c_n-d_n \iff$ $a_{n+1}+c_{n+1}=2(a_n+c_n),n\in\mathbb{N}$ and, therefore, $a_n+c_n=2^{n-1}(a_1+c_1)\iff a_n+c_n=2^{n-1}(a+c-b-d),n\in\mathbb{N}$.
Hence, $b_n+d_n=-(a_n+c_n)=2^{n-1}(b+d-a-c),n\in\mathbb{N}$.

Noting that $\max\{a_n,c_n\}\geq\dfrac{a_n+c_n}{2}=2^{n-2}(a+c-b-d)$,

$$\max\{b_n,d_n\}\geq\frac{b_n+d_n}{2}=2^{n-2}(b+d-a-c)$$

we obtain $\max\{a_n,c_n,b_n,d_n\}\geq 2^{n-2}|a+c-b-d|$.

(Because $\begin{cases}x\geq p\\ y\geq -p\end{cases}\implies\max\{x,y\}\geq|p|$.

Indeed, if $x\geq y$ then $\begin{cases}x\geq p\\ x\geq -p\end{cases}\iff x\geq|p|\iff\max\{x,y\}\geq|p|$

and if $x<y$ then $\begin{cases}y\geq p\\ y\geq -p\end{cases}\iff y\geq|p|\iff\max\{x,y\}\geq|p|$).

Thus, in case $a+c\neq b+d$, since $|a+c-b-d|\geq 1$ we obtain

$$\mu_n:=\max\{a_n,b_n,c_n,d_n\}\geq 2^{n-2}\geq 2^{\frac{n-2}{2}}\quad\text{for } n\geq 2.$$

Remark.

In fact inequality $\mu_n\geq 2^{\frac{n-2}{2}}$ for $n\geq 2$ holds any $n\in\mathbb{N}$.

For $n=1$ we have $\max\{a_1,b_1,c_1,d_1\}=\max\{a-b,b-c,c-d,d-a\}\geq 1>2^{\frac{1-2}{2}}=\dfrac{1}{\sqrt{2}}$.Indeed, since at least one of difference of integers $a-b,b-c,c-d,d-a$ isn't zero and $a-b+b-c+c-d+d-a=0$ then can't be $a-b\leq 0,b-c\leq 0,c-d\leq 0,d-a\leq 0$. (othervice if $a-b\leq 0,b-c\leq 0,c-d\leq 0,d-a\leq 0$ then $a-b=b-c=c-d=c-a=0$). Hence, at least one of $a-b,b-c,c-d,d-a$ begger then zero and, therefore, $\max,\{a-b,b-c,c-d,d-a\}\geq 1$.

Consider now case $a+c=b+d$. Due to cyclic symmetry of recurrence **(1)** we can assume that $a\neq 0$ or $b\neq 0$ because if $a=b=0$ then $c=d$ and at least one of them isn't zero (otherwice, we obtain $a=b=c=d$ and that is contradict to condition of the problem). In that case we can cyclicly rename numbers and get $a \neq 0$ or $b \neq 0$.Then $a_n + c_n = b_n + d_n = 0, n \in \mathbb{N}$. It can be easy proved by Math Induction.Indeed, we have $a_1 + c_1 = a - b + c - d = 0$ and $b_1 + d_1 = -(a_1 + c_1) = 0$ and since $a_n + c_n = b_n + d_n = 0$ then $a_n + c_n = 0$ yelds $a_{n+1} + c_{n+1} = a_n - b_n + c_n - d_n = 0$ and $b_{n+1} + d_{n+1} = -(a_{n+1} + b_{n+1}) = 0$.

Since $a_{n+1} + b_{n+1} = a_n - b_n + b_n - c_n = a_n - c_n = 2a_n$ and $a_{n+2} = a_{n+1} - b_{n+1}$ then

$$2a_n + a_{n+2} = a_{n+1} + b_{n+1} + a_{n+1} - b_{n+1} \iff a_{n+2} = 2a_{n+1} - 2a_n, n \in \mathbb{N} \cup \{0\} \iff$$

$$\frac{a_{n+2}}{\left(\sqrt{2}\right)^{n+2}} - 2 \cdot \frac{1}{\sqrt{2}} \frac{a_{n+1}}{\left(\sqrt{2}\right)^{n+1}} + 2 \cdot \frac{a_n}{2 \cdot \left(\sqrt{2}\right)^n} = 0 \iff$$

(1) $x_{n+2} - 2\cos\frac{\pi}{4} \cdot x_{n+1} + x_n = 0, n \in \mathbb{N} \cup \{0\}$, where $x_n := \dfrac{a_n}{\left(\sqrt{2}\right)^n}, n \in \mathbb{N} \cup \{0\}$.

Since general solution of the homogeneous recurrence **(1)** is $x_n = \alpha \cos\frac{n\pi}{4} + \beta \sin\frac{n\pi}{4}$, $n \in \mathbb{N} \cup \{0\}$ then $a_n = \left(\sqrt{2}\right)^n \left(\alpha \cos\frac{n\pi}{4} + \beta \sin\frac{n\pi}{4}\right), n \in \mathbb{N} \cup \{0\}$,where α, β be some real constants. Since $a_0 = \left(\sqrt{2}\right)^0 \left(\alpha \cos\frac{0 \cdot \pi}{4} + \beta \sin\frac{0 \cdot \pi}{4}\right) \iff a = \alpha$ and $a_1 = \left(\sqrt{2}\right)^1 \left(\alpha \cos\frac{1 \cdot \pi}{4} + \beta \sin\frac{1 \cdot \pi}{4}\right) \iff a_1 = \sqrt{2}\left(a \cdot \frac{1}{\sqrt{2}} + \beta \cdot \frac{1}{\sqrt{2}}\right) \iff a - b = a + \beta \iff \beta = -b$ then $a_n = \left(\sqrt{2}\right)^n \left(a \cos\frac{n\pi}{4} - b \sin\frac{n\pi}{4}\right), n \in \mathbb{N} \cup \{0\}$.

Hence, $\max\{a_n, c_n\} = \max\{a_n, -a_n\} = |a_n| = 2^{n/2}\left|a \cos\frac{n\pi}{4} - b \sin\frac{n\pi}{4}\right|, n \in \mathbb{N}$ and, cyclic we have

$$\max\{b_n, d_n\} = 2^{n/2}\left|b \cos\frac{n\pi}{4} - c \sin\frac{n\pi}{4}\right| = 2^{n/2}\left|b \cos\frac{n\pi}{4} + a \sin\frac{n\pi}{4}\right|, n \in \mathbb{N}.$$

Let $\alpha_n := \left|a \cos\frac{n\pi}{4} - b \sin\frac{n\pi}{4}\right|, \beta_n := \left|b \cos\frac{n\pi}{4} + a \sin\frac{n\pi}{4}\right|$.

Then, since $\max\{x, y\} = \dfrac{x + y + |x - y|}{2}$ we obtain

$$\max\{a_n, b_n, c_n, d_n\} = 2^{n/2} \max\{\alpha_n, \beta_n\} = 2^{\frac{n-2}{2}}\left(\alpha_n + \beta_n + |\alpha_n - \beta_n|\right) \ge 2^{\frac{n-2}{2}}\left(\alpha_n + \beta_n\right).$$

Therefore,

$$\max\{a_n, b_n, c_n, d_n\} \ge 2^{\frac{n-2}{2}} \max\left\{\left|a \cos\frac{n\pi}{4} - b \sin\frac{n\pi}{4}\right| + \left|b \cos\frac{n\pi}{4} + a \sin\frac{n\pi}{4}\right|\right\}$$

.

Let $\cos\varphi := \dfrac{a}{\sqrt{a^2 + b^2}}, \sin\varphi := \dfrac{b}{\sqrt{a^2 + b^2}}$ then $\alpha_n + \beta_n = \sqrt{a^2 + b^2}\left(\left|\cos\left(\varphi + \frac{n\pi}{4}\right)\right| + \left|\sin\left(\varphi + \frac{n\pi}{4}\right)\right|\right) \ge \sqrt{a^2 + b^2}\left(\sin^2\left(\varphi + \frac{n\pi}{4}\right) + \cos^2\left(\varphi + \frac{n\pi}{4}\right)\right) \ge \sqrt{a^2 + b^2} \ge 1$ because $a \neq$

0 or $b \neq 0$. Thus, in the case $a + c = b + d$ for any $n \in \mathbb{N}$ holds inequality $\max\{a_n, b_n, c_n, d_n\} \geq 2^{\frac{n-2}{2}}$.

Problem 6.6(19-Met. Rec.)
Recurrence that define the sequence $a_1, a_2, ..., a_n, ...$ can be rewritten as
(1) $\qquad a^n = \sum_{t \in D(n)} a_t,$
where $a_1 = a$ and $D(n)$ is set of all natural divisors of n.

First we collect experimental material which can clarify place of this problem among the known facts.

i. L et/ $n = p$,where p is prime number. Then $D(p) = \{1, p\}$ and we get $a^p = a_1 + a_p \iff a_p = a^p - a \vdots p$ by Little Fermat Theorem.

ii. Let/ $n = p^2$, where p is prime number. Then $D(p^2) = \{1, p, p^2\}$ and we get $a^{p^2} = a_1 + a_p + a_{p^2} \iff a^{p2} = a + a^p - a + a_{p^2} \iff a_{p^2} = a^{p^2} - a^p$.

If $\gcd(a, p) = 1$ then $a_{p^2} = a^p\left(a^{p^2 - p} - 1\right)$.Since $p^2 - p = \varphi(p^2)$,where $\varphi(a)$ is Euler's totient function (that counts the natural numbers which does not exceed a and relatively prime with a) then by Euler Theorem $a^{\varphi(p^2)} - 1 \vdots p^2$ and, therefore, $a_{p^2} \vdots p^2$.If $\gcd(a, p) \neq 1$ then $a \vdots p$ and $a^p \vdots p^2 \implies a_{p^2} \vdots p^2$ since $p \geq 2$. Thus $a_{p^2} \vdots p^2$ for any $a > 1$.

iii. Let/ $n = p^k$ then by Math induction we will prove that $a_{p^k} = a^{p^k} - a^{p^{k-1}}$.
Base of Math Induction we already have.
For $n = p^{k+1}$ we have

$$a^{p^{k+1}} = \sum_{i=0}^{k} a_{p^i} + a_{p^{k+1}} = a + \sum_{i=1}^{k}\left(a^{p^i} - a^{p^{i-1}}\right) + a_{p^{k+1}} = a^{p^k} a + a_{p^{k+1}}.$$

Hence, $a_{p^{k+1}} = a^{p^{k+1}} - a^{p^k}$. Thus, $a_{p^k} = a^{p^{k-1}}\left(a^{p^k - p^{k-1}} - 1\right) = a^{p^{k-1}}\left(a^{\varphi(p^k)} - 1\right)$ and further as in **ii.**

If $\gcd(a, p) = 1$ then $a^{\varphi(p^k)} - 1 \vdots p^k \implies a_{p^k} \vdots p^k$;

If $\gcd(a, p) \neq 1$ then $a \vdots p$ and $a^{p^{k-1}} \vdots p^{p^{k-1}}$.Since for any $k \in \mathbb{N}$ holds*

$p^{k-1} \geq 2^{k-1} \geq k$ then $a^{p^{k-1}} \vdots p^k \implies a_{p^k} \vdots p^k$.

(* We have $2^{k-1} = k$ for $k = 1, 2$ and for any $k \geq 3$
from $2^{k-1} > k$ follows $2^k = 2 \cdot 2^{k-1} > 2k > k + 1$).

iX. Let $n = p \cdot q$,where p, q be different prime numbers. Then
$a^{pq} = a_1 + a_p + a_q + a_{pq} = a + a^p - a + a^q - a + a_{pq} \iff a_{pq} = a^{pq} - a^p - a^q + a$

and we have $a^{pq} - a^p - a^q + a = a^p\left(a^{p(q-1)} - 1\right) - a\left(a^{q-1} - 1\right) \vdots \left(a^{q-1} - 1\right)$ because

$a^{p(q-1)} - 1 = \left(a^{q-1}\right)^p - 1 \vdots \left(a^{q-1} - 1\right)$ and similarly

$a^{pq} - a^p - a^q + a = a^q\left(a^{q(p-1)} - 1\right) - a\left(a^{p-1} - 1\right) \vdots \left(a^{p-1} - 1\right)$.

If $\gcd(a,pq)=1$ then by Little Fermat Theorem $a^{p-1}-1\,\vdots\,p, a^{q-1}-1\,\vdots\,q$ and, therefore,

$$\begin{cases} a_{pq}\,\vdots\,p \\ a_{pq}\,\vdots\,q \end{cases} \implies a_{pq}\,\vdots\,pq \text{ (since } \gcd(p,q)=1);$$

If $\gcd(a,pq)\neq 1$ then by cosideration cases $\gcd(a,p)\neq 1$ and $\gcd(a,q)=1$ or $\gcd(a,p)=1$ and $\gcd(a,q)\neq 1$, or $\gcd(a,p)\neq 1$ and $\gcd(a,q)\neq 1$ we again, as we did before, obtain $a_{pq}\,\vdots\,pq$.

We stop consideration of particular cases and and proceed to the problem in general case using Math Induction by $n\in\mathbb{N}\setminus\{1\}$ (because for $n=1$ statement of the problem is trivial)

1. Base of induction for $n=2$ already proved in **i**. when $p=2$.

2. Step of Math Induction.

For any $n>2$ assume that $a_k\,\vdots\,k$ for all $k<n$.Then in particular $a_k\,\vdots\,k$ for all $k\in D(n)\setminus\{n\}$.

Let p be prime divisor on n and $k:=ord_p n=\max\left\{t\mid t\in\mathbb{N}\cup\{0\} \text{ and } n\,\vdots\,p^t\right\}$.

Then $n=p^k m$ and $\gcd(m,p)=1$.Let d is any divisor of n, then $d=p^i t$,where $0\leq i\leq k$, t is divisor of m and $p^i D(m):=\left\{p^i t\mid t\in D(m)\right\}$. Since $D(n)=\bigcup\limits_{i=0}^{k}p^i D(m)$ we can rewrite **(1)** as $a_n=\sum\limits_{i=0}^{k}\sum\limits_{t\in D(m)}a_{p^i t}=$

$$\sum_{i=0}^{k-1}\sum_{t\in D(m)}a_{p^i t}+\sum_{t\in D(m)}a_{p^k t}=\sum_{i=0}^{k-1}\sum_{t\in D(m)}a_{p^i t}+a_{p^k m}+\sum_{t\in D(m)\setminus\{m\}}a_{p^k t}=$$

$$\sum_{t\in D(m\cdot p^{k-1})}a_{p^i t}+a_{p^k m}+\sum_{t\in D(m)\setminus\{m\}}a_{p^k t}=a^{p^{k-1}m}+a_n+\sum_{t\in D(m)\setminus\{m\}}a_{p^k t}$$

(because $p^k m=n$ and $\sum\limits_{t\in D(m\cdot p^{k-1})}a_{p^i t}=a^{p^{k-1}m}$ by **(1)**).

Since $t<m$ then $p^k t<n$ and by supposition of Math Induction $a_{p^k t}\,\vdots\,p^k t$. It yields $a_{p^k t}\,\vdots\,p^k$ and, therefore, $a_n\equiv a^n-a^{p^{k-1}m}\left(\mathrm{mod}\,p^k\right)=a^{p^{k-1}m}\left(a^{m\left(p^k-p^{k-1}\right)}-1\right)\left(\mathrm{mod}\,p^k\right)=b^{p^{k-1}}\left(b^{\varphi\left(p^k\right)}-1\right)\left(\mathrm{mod}\,p^k\right)$, where $b:=a^m$ for short. As above we consider two cases:

1. $\gcd(a,p)=1\implies\gcd(b,p)=1$ and then by Euler's Theorem

$$b^{\varphi\left(p^k\right)}-1\,\vdots\,p^k\iff a^{m\left(p^k-p^{k-1}\right)}-1\equiv 0\left(\mathrm{mod}\,p^k\right)\implies a_n\equiv 0\left(\mathrm{mod}\,p^k\right)$$

2. $\gcd(a,p)\neq 1\iff a\,\vdots\,p$. Suffice to note that $a^{p^{k-1}}\,\vdots\,p^k$ because $p^{k-1}\geq 2^{k-1}\geq k$ (see the similar case in **iii.**). And again $a_n\equiv 0\left(\mathrm{mod}\,p^k\right)$.

Since $n=p_1^{k_1}p_2^{k_2}...p_l^{p_l}$ (prime deciomposition of n) and $a_n\,\vdots\,p_i^{k_i}, i=1,2,...,l$ then $a_n\,\vdots\,n$.

Remark.

We will clarify the origin of recurrence, represented in the problem and at the same time we can give another combinatorial solution of it.

Let $n \in \mathbb{N},\ I_n := \{1, 2, ..., n\}, R_a := \{0, 1, ..., a-1\}$ and let $PM(n, a)$ be set of all periodic functions from $\mathbb{Z}$ to R_a with period n .

Let $f \in PM(n, a)$ .Then $f(m + kn) = f(m)$ for any $m, k \in \mathbb{Z}$. Indeed, since $f(m + n) = f(m)$ for any $m \in \mathbb{Z}$ then $f(m) = f((m - n) + n) = f(m - n)$ and by Math Induction easy to prove $f(m \pm kn) = f(m)$ for any $k \in \mathbb{N}$. Since for any $m \in \mathbb{Z}$ we have unique representation $m = kn + r, k \in \mathbb{Z}, r \in I_n$ (see **Remark** to **Problem 5.6**) then $f(m) = f(r + kn) = f(r)$. Thus, any function $f \in PM(n, a)$ is completely determined by it's restriction on I_n.

Since we have exactly a^n different functions from I_n to R_a then $|PM(n, a)| = a^n$.For each $f \in PM(n, a)$ we denote $p(f)$ smallest natural period of f, that is $p(f) = k$ if $f(m + k) = f(m)$ for any $m \in \mathbb{Z}$ and for any $1 \le i < k$ there is $m \in \mathbb{Z}$ such that $f(m + i) \neq f(m)$. Obvious that if n is period of f then n is multiple of $p(f)$ that is $n \,\vdots\, p(f)$ .We will say that $p(f)$ is main period of f. For any $k \in D(n)$ we denote $F_k := \{f \mid f \in PM(n, a) \text{ and } p(f) = k\}$ .Let $a_k = |F_k|$ be number of $n-$periodic functions from $\mathbb{Z}$ to R_a with main period k.In particular, F_n is the set of all periodic functions with main period n and $|F_n| = a_n$. Obvious that $a_1 = |F_1|$ because we have only a functions from $I_1 = \{1\}$ to R_a.

Since $PM(n, a) = \bigcup\limits_{k \in D(n)} F_k$ and $F_{k_1} \cap F_{k_2} = \varnothing$ if $k_1 \neq k_2$ then

$$|PM(n, a)| = \sum_{k \in D(n)} |F_k| \iff a^n = \sum_{k \in D(n)} a_k \iff a^n = a_n + \sum_{k \in D(n), k < n} a_k \iff$$

$$a_n = a^n - \sum_{k \in D(n), k < n} a_k.$$

Now we wil prove that a_n divisible by n. Let $S : F_n \longrightarrow F_n$ be 1-step shift operator, that is $S(f)(m) = f(m + 1)$ and let $S^{i+1}(f) := S^i(S(f))$ .Then $S^i(f)(m) = f(m + i)$ and $p(S^m(f)) = p(f)$ .By definition $S^0(f) := f$ and obvious that $S^n(f) = f$. Consider the following equivalence relation on F_n :

Two functions $f, g \in F_n$ is equivalent if there are i and j such that $S^i(f) = S^j(g)$.

Then for any $f \in F_n$ set

$$\mathcal{O}(f) := \left\{f, S^1(f), S^2(f), ..., S^{n-1}(f)\right\}$$

is class of equivalency of f with respect to defined above equivalence relation on F_n. Note that $|\mathcal{O}(f)| = n$. Indeed, $S^i(f) \neq S^j(f)$ for $0 \le i < j < n$ becuse if we assume that $S^i(f) = S^j(f)$ then for any $m \in \mathbb{Z}$ we have $f(m) = f((m - i) + j) = f(m + (j - i))$. Hence, $0 < j - i < n$ and $j - i$ is a period of f , that is the contradiction with $p(f) = n$. Let F be set of representators of classes of equivalency (by one function from each class of equivalency). Then $\mathcal{O}(f_1) \cap O(f_2) = \varnothing$ if $f_1, f_2 \in F$ and $f_1 \neq f_2$. Since $F_n = \bigcup_{f \in F} \mathcal{O}(f)$ and $|\mathcal{O}(f)| = n$ for any $f \in F_n$ then

$$|F_n| = \sum_{f \in F} |\mathcal{O}(f)| = n|F| \vdots n \iff a_n \vdots n.$$

Problem 6.7*

Since

(1) $a_{n+2} = a_{n+1}a_n - 2(a_{n+1} + a_n) - a_{n-1} + 8 \iff$
$a_{n+2} - 2 = (a_{n+1} - 2)(a_n - 2) - (a_{n-1} - 2)$

then for $b_n := a_n - 2$ we obtain recurrence

(2) $b_{n+2} = b_{n+1}b_n - b_{n-1}, n \in \mathbb{N}$ with $b_0 = 2, b_1 = b_2 = \left(a^2 - 2\right)^2 - 2.$

For further we need

Lemma.

Let sequense $(P_n)_{n\geq 0}$ be determined by the recurrence

(3) $P_{n+2} = P_{n+1}P_n - P_{n-1}, n \in \mathbb{N}$ with $P_0 = 2, P_1 = P_2 = x > 2,$

and let (f_n) be sequence of Fibonacci numbers ($f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}$ and $f_0 = 1,\ f_1 = 1$).

Then requrrence **(3)** determine polynomial $P_n(x)$ of x, of degree f_n with integer coefficients, such that $P_n(\cosh(t)) = 2\cosh(f_n t)$.

Proof.

Since

$$2\cosh(f_0 t) = 2\cosh(0) = 2, 2\cosh(f_1 t) = 2\cosh(f_2 t) =$$

$$2\cosh t = x \quad and 2\cosh(f_{n+1}t) \cdot 2\cosh(f_n t) - 2\cosh(f_{n-t}t) =$$

$$4\cosh(f_{n+1}t)\cosh(f_n t) - 2\cosh(f_{n-t}t) = 2(\cosh(f_{n+1}t + f_n t) + \cosh(f_{n+1}t - f_n t)) - 2\cosh(f_{n-t}t) =$$

$$2\cosh(f_{n+2}t) + 2\cosh(f_{n-1}t) - 2\cosh(f_{n-t}t) = 2\cosh(f_{n+2}t)$$

then by Math Induction we obtain that $P_n(x) = 2\cosh\left(f_{n+2} \cdot \cosh^{-1}\left(\frac{x}{2}\right)\right)$

Coming back to recurrence **(2)** and denoting $t := \cosh^{-1}\left(\frac{a}{2}\right)$ we obtain that $\left(k^2 - 2\right)^2 - 2 = \left(4\cosh^2 t - 2\right)^2 - 2 = 4\left(2\cosh^2 t - 1\right)^2 - 2 = 4\cosh^2 2t - 2 = 2\left(2\cosh^2 2t - 1\right) = 2\cosh 4t$ and then accordingly to Lemma $b_n = 2\cosh(4 f_n t)$ .Therefore, $a_n = 2\cosh(4 f_n t) + 2 = 4\cosh^2(2 f_n t)$ and since $\cosh(x) > 0$ for any x then $2 + \sqrt{a_n} = 2 + 2\cosh(2 f_n t) = 2(1 + \cosh(2 f_n t)) = 4\cosh^2(f_n t) = (2\cosh(f_n t))^2 = (P_n(a))^2$.

Problem 6.8.

a) Let $t_n := \sqrt{1+3a_n}$ then $t_1 = 3, a_n = \dfrac{t_n^2-1}{3}$ and, therefore,

$$\frac{t_{n+1}^2-1}{3} = \frac{1}{27}\left(8+3\cdot\frac{t_n^2-1}{3}+8t_n\right) \iff$$

$$t_{n+1}^2 = \frac{1}{9}\left(16+8t_n+t_n^2\right) = \left(\frac{t_n+4}{3}\right)^2 \iff t_{n+1} = \frac{t_n+4}{3}\ since t_n > 0.$$

Thus we have $3^{n+1}t_{n+1} = 3^n t_n + 4\cdot 3^n \iff 3^{n+1}t_{n+1} = 3^n t_n + 2\left(3^{n+1}-3^n\right) \iff$

$3^{n+1}t_{n+1} - 2\cdot 3^{n+1} = 3^n t_n - 2\cdot 3^n, n\in\mathbb{N} \iff 3^n t_n - 2\cdot 3^n = 3^1 t_1 - 2\cdot 3^1 \iff$
$3^n t_n - 2\cdot 3^n = 3 \iff t_n = \dfrac{2\cdot 3^n+3}{3^n} = \dfrac{2\cdot 3^{n-1}+1}{3^{n-1}}.$

Hence, $a_n = \dfrac{t_n^2-1}{3} = \dfrac{\left(\dfrac{2\cdot 3^{n-1}+1}{3^{n-1}}\right)^2-1}{3} = \dfrac{3^{2n-1}+4\cdot 3^{n-1}+1}{3^{2n-1}}.$

b) Let $t_n := \sqrt{1+24a_n}$ then $t_1 = 5, a_n = \dfrac{t_n^2-1}{24}$ and, therefore,

$\dfrac{t_{n+1}^2-1}{24} = \dfrac{1}{16}\left(1+4\cdot\dfrac{t_n^2-1}{24}+t_n\right) \iff$

$t_{n+1}^2 = \dfrac{24}{16}\left(1+4\cdot\dfrac{t_n^2-1}{24}+t_n\right)+1 = \dfrac{1}{4}\left(t_n+3\right)^2.$

Since $t_n > 0$ then $t_{n+1} = \dfrac{t_n+3}{2} \iff 2^{n+1}t_{n+1} = 2^n t_n + 3\cdot 2^n \iff$

$$2^{n+1}t_{n+1} = 2^n t_n + 3\left(2^{n+1}-2\right)^n \iff 2^{n+1}t_{n+1} - 3\cdot 2^n = 2^n t_n - 3\cdot 2^n, n\in\mathbb{N} \iff$$

$$2^n t_n - 3\cdot 2^n = 2^1 t_1 - 3\cdot 2^1 \iff 2^n t_n - 3\cdot 2^n = 4 \iff t_n = \frac{3\cdot 2^n+4}{2^n} = \frac{3\cdot 2^{n-2}+1}{2^{n-2}}.$$

Hence, $a_n = \dfrac{t_n^2-1}{24} = \dfrac{\left(\dfrac{3\cdot 2^{n-2}+1}{2^{n-2}}\right)^2-1}{24} = \dfrac{2^{2n-1}+3\cdot 2^{n-1}+1}{3\cdot 2^{2n-1}}.$

Remark.
Note that $2^{2n-1}+3\cdot 2^{n-1}+1 \equiv 0 \pmod 3$.

Problem 6.9.
a),b) $\sqrt{a_{n+1}+1}-\sqrt{a_{n+1}} = \left(\sqrt{2}-1\right)^{n+1} = \left(\sqrt{2}-1\right)\left(\sqrt{2}-1\right)^n =$
$\left(\sqrt{2}-1\right)\left(\sqrt{a_n+1}-\sqrt{a_n}\right) \iff \sqrt{2\left(a_n+1\right)}+\sqrt{a_n}-\left(\sqrt{2a_n}+\sqrt{a_n+1}\right).$
Since, $\left(\sqrt{2\left(a_n+1\right)}+\sqrt{a_n}\right)^2 = 3a_n+2+2\sqrt{2a_n\left(a_n+1\right)} = 3a_n+2+2t_n$
and $\left(\sqrt{2a_n}+\sqrt{a_n+1}\right)^2 = 3a_n+1+2t_n$, where $t_n := \sqrt{2a_n\left(a_n+1\right)}$
we obtain:

$$\sqrt{2(a_n+1)}+\sqrt{a_n}=\sqrt{3a_n+2+2t_n},\ \sqrt{2a_n}+\sqrt{a_n+1}=\sqrt{3a_n+1+2t_n}$$

.

Hence*,

$$\sqrt{a_{n+1}+1}-\sqrt{a_{n+1}}=\sqrt{(3a_n+1+t_n)+1}-\sqrt{3a_n+1+2t_n} \implies a_{n+1}=3a_n+2t_n+1.$$

(*Since function $h(x)=\sqrt{x+1}-\sqrt{x}=\dfrac{1}{\sqrt{x+1}+\sqrt{x}}$ strictly monotone decrease, then from $h(x_1)=h(x_2)$ follows $x_1=x_2$).

From the other hand

$$t_{n+1}^2=2a_{n+1}(a_{n+1}+1)=2(3a_n+2t_n+1)^2+6a_n+4t_n+2=$$

$$18a_n^2+8t_n^2+2+24a_nt_n+8t_n+12a_n+6a_n+4t_n+2=18a_n^2+24a_nt_n+8t_n^2+18a_n+12t_n+4=$$

$$16a_n^2+24a_nt_n+16a_n+\left(2a_n^2+2a_n\right)+8t_n^2+12t_n+4=$$

$$16a_n^2+9t_n^2+4+24a_nt_n+16a_n+12t_n=(4a_n+3t_n+2)^2 \iff t_{n+1}=4a_n+3t_n+2.$$

So, we obtain system of recurrences:

$$\begin{cases} a_{n+1}=3a_n+2t_n+1 \\ t_{n+1}=4a_n+3t_n+2 \end{cases},n\in\mathbb{N}\ .$$

Hereof $2t_n=a_{n+1}-3a_n-1 \implies 2t_{n+1}=a_{n+2}-3a_{n+1}-1$.Since $2t_{n+1}=8a_n+6t_n+4$, then $a_{n+2}-3a_{n+1}-1=8a_n+3a_{n+1}-9a_n-3+4 \iff a_{n+2}-6a_{n+1}+a_n=2$.

Since $4a_n=t_{n+1}-3t_n-2 \implies 4a_{n+1}=t_{n+2}-3t_{n+1}-2$ and $4a_{n+1}=12a_n+8t_n+4$ we obtain $t_{n+2}-3t_{n+1}-2=3t_{n+1}-9t_n-6+8t_n+4 \iff t_{n+2}-6t_{n+1}+t_n=6$. Initial condition follows from identities:

From $\sqrt{2}-1=\sqrt{a_1+1}-\sqrt{a_1}$ we obtain $a_1=1$;

From $\left(\sqrt{2}-1\right)^2=3-2\sqrt{2}=\sqrt{9}-\sqrt{8}$ we obtain $a_2=8$.

Since $t_n=\sqrt{2a_n(a_n+1)}$ and $a_1=1,a_2=8$ we obtain $t_1=2,t_2=12$.So, from recurrences $a_{n+2}-6a_{n+1}+a_n=2$ with $a_1=1,a_2=8$ and $t_{n+2}-6t_{n+1}+t_n=6$ with $t_1=2,t_2=12$ follows that a_n and t_n are integers for all $n\in\mathbb{N}$.

Problem 6.10.

Using substitution $a_n:=b_n+1$ we obtain $b_0=b_1=1$ and

$$b_{n+2}+1=\frac{2b_{n+1}+2-3b_{n+1}b_n-3b_{n+1}-3b_n-3+17b_n+17-16}{3b_{n+1}+3-4b_{n+1}b_n-4b_{n+1}-4b_n-4+18b_n+18-17} \iff$$

$$b_{n+2}=\frac{-3b_{n+1}b_n-b_{n+1}+14b_n}{-4b_{n+1}b_n-b_{n+1}+14b_n}-1=\frac{b_{n+1}b_n}{-4b_{n+1}b_n-b_{n+1}+14b_n}\iff$$

$$\frac{1}{b_{n+2}}=\frac{14}{b_{n+1}}-\frac{1}{b_n}-4.$$

Thus, for sequence $c_n:=\dfrac{1}{b_n}$ we have recurrence

$$c_{n+2}-14c_{n+1}+c_n=-4, n\in\mathbb{N}\cup\{0\} \text{ with } c_0=c_1=1,$$

and original problem equivalently reduced to the problem:

Prove that c_n for any $n\in\mathbb{N}\cup\{0\}$ is a perfect square of natural number.

There is two ways to solve this problem.

First way(use standard technic of solving second degree linear recurrence with constant coefficients)**:**

Since $c_{n+2}-14c_{n+1}+c_n=-4 \iff \left(c_{n+2}-\dfrac{1}{3}\right)-14\left(c_{n+1}-\dfrac{1}{3}\right)+\left(c_n-\dfrac{1}{3}\right)=0$ then $c_n-\dfrac{1}{3}=\alpha\left(7+2\sqrt{3}\right)^n+\beta\left(7-2\sqrt{3}\right)^n=\alpha\left(2+\sqrt{3}\right)^{2n}+\beta\left(2-\sqrt{3}\right)^{2n}$, $n\in\mathbb{N}\cup\{0\}$ where $7+2\sqrt{3}$ and $7-2\sqrt{3}$ are the roots of quadratic equation $\chi^2-14\chi+1=0$,associated with recurrence $x_{n+2}-14x_{n+1}+x_n=0$.

From initial conditions $c_0-\dfrac{1}{3}=c_1-\dfrac{1}{3}=\dfrac{2}{3}$ we obtain $\alpha=\dfrac{2-\sqrt{3}}{6}=\left(\dfrac{\sqrt{3}-1}{2\sqrt{3}}\right)^2$ and $\beta=\dfrac{2+\sqrt{3}}{6}=\left(\dfrac{\sqrt{3}+1}{2\sqrt{3}}\right)^2$. Since $\dfrac{\sqrt{3}-1}{2\sqrt{3}}\cdot\dfrac{\sqrt{3}+1}{2\sqrt{3}}=\dfrac{1}{3}$ then

$$c_n=\left(\frac{\left(\sqrt{3}-1\right)\left(2+\sqrt{3}\right)^n}{2\sqrt{3}}\right)^2+\left(\frac{\left(\sqrt{3}+1\right)\left(2-\sqrt{3}\right)^n}{2\sqrt{3}}\right)^2+ 2\cdot\frac{\left(\sqrt{3}-1\right)\left(2+\sqrt{3}\right)^n}{2\sqrt{3}}\cdot\frac{\left(\sqrt{3}+1\right)\left(2-\sqrt{3}\right)^n}{2\sqrt{3}}=d_n^2,$$

where $d_n:=\dfrac{\left(\sqrt{3}-1\right)\left(2+\sqrt{3}\right)^n}{2\sqrt{3}}+\dfrac{\left(\sqrt{3}+1\right)\left(2-\sqrt{3}\right)^n}{2\sqrt{3}}$

is integer for any $n\in\mathbb{N}\cup\{0\}$ because d_n satisfy to recurrence

$d_{n+2}-4d_{n+1}+d_n=0$ and $d_0=d_1=1$.

Second way:

First note that $c_n\geq 0$ for any $n\in\mathbb{N}\cup\{0\}$. Really, since $c_0=c_1=1$ then rewriting recurrence for c_n in the form $c_{n+1}-c_n=c_n-c_{n-1}+12c_n-4$, $n\in\mathbb{N}$, and using Math. Induction we conclude that $c_n-c_{n-1}\geq 0, n\in\mathbb{N}$.

Hence, $c_n\geq c_0=1$.Denote $d_n:=\sqrt{c_n}$ and, in supposition that d_n satisfy to the recurrence $d_{n+1}-pd_n+d_{n-1}=0$, $n\in\mathbb{N}$ with $d_0=d_1=1$, we will find recurrence for d_n^2 . Since $d_{n+1}d_{n-1}-d_n^2=\left(pd_n-d_{n-1}\right)d_{n-1}-d_n\left(pd_{n-1}-d_{n-2}\right)=d_nd_{n-2}-d_n^2$, then $d_{n+1}d_{n-1}-d_n^2=d_2d_0-d_1^2=pd_1d_0-d_0^2-d_1^2=p-2$ and from the other hand $p^2d_n^2=d_{n+1}^2+2d_{n+1}d_{n-1}+d_{n-1}^2$.Thus, $p^2d_n^2=d_{n+1}^2+2\left(p-2+d_n^2\right)+d_{n-1}^2 \iff d_{n+1}^2-\left(p^2-2\right)d_n^2+d_{n-1}^2=4-2p$.From claim $p^2-2=14$ and $4-2p=-4$ we obtain $p=4$.Since d_n^2 and c_n satisfy to the same recurrence and to the same initial conditions then $c_n=d_n^2$.

Problem 6.11*.

First we will find for any $n, m \in \mathbb{N} \cup \{0\}$ representation of a_{n+m} as linear combination of a_n, a_{n+1},that is $a_{n+m} = p_m a_n + q_m a_{n+1}$ where coefficients p_m, q_m we need to find. Note that $a_{n+0} = p_0 a_n + q_0 a_{n+1} \implies p_0 = 1, q_0 = 0$, $a_{n+1} = p_1 a_n + q_1 a_{n+1} \implies p_1 = 0, q_1 = 1$. Also we have $a_{n+m+1} = 2a_{n+m} + a_{n+m-1} \iff p_{m+1}a_n + q_{m+1}a_{n-1} = 2(p_m a_n + q_m a_{n-1}) + 2(p_{m-1}a_n + q_{m-1}a_{n-1}), n \in \mathbb{N} \cup \{0\} \implies p_{m+1} = 2p_m + p_{m-1}$ and $q_{m+1} = 2q_m + q_{m-1}$. Since $a_{-1} = a_1 - 2a_0 = 1, q_0 = 0, q_1 = 1$ and $p_0 = 1, p_1 = 0$ we obtain $q_m = a_m, p_m = a_{m-1},\ m \in \mathbb{N} \cup \{0\}$.

Thus, $a_{n+m} = a_{m-1}a_n + a_m a_{n+1}$ for any $n, m \in \mathbb{N} \cup \{0\}$ and, in particular, $a_{2n} = a_{n-1}a_n + a_n a_{n+1} = a_n(a_{n-1} + a_{n+1}) = a_n(a_{n-1} + (2a_n + a_{n-1})) = 2a_n(a_n + a_{n-1}), n \in \mathbb{N} \cup \{0\}$ .Since $b_n := a_n + a_{n-1}$ satisfy to recurrence $b_{n+1} = 2b_n + b_{n-1}, n \in \mathbb{N}$ and $b_0 = b_1 = 1$ then $b_{n+1} \equiv b_{n-1} \pmod 2$ implies $b_n \equiv 1 \pmod 2$ .Thus, $a_{2n} = 2a_n b_n$ and by Math Induction we obtain $a_{2^k n} = 2^k a_n c_k,\ k \in \mathbb{N}$ where c_k is some odd number. Indeed, $a_{2n} = 2a_n c_1$,($c_1 := b_n$)and for any $k \in \mathbb{N}$ assuming $a_{2^k n} = 2^k a_n c_k, c_k \equiv 1 \pmod 2$ we obtain $a_{2^{k+1}n} = 2a_{2^k n} b_{2^k n} = 2 \cdot 2^k a_n c_k b_{2^k n} = 2^{k+1} a_n c_{k+1}$ where $c_{k+1} = c_k b_{2^k n} \equiv 1 \pmod 2$.

Let m be any odd natural then a_m is odd as well because $a_1 = 1$ and $a_m \equiv a_{m-2} \pmod 2$ implies $a_m \equiv 1 \pmod 2$. Then $a_{2^k m} = 2^k a_m c_k$ for any $k \in \mathbb{N} \cup \{0\}$ and any odd natural m and, therefore, $a_n \vdots 2^k \iff n \vdots 2^k$.

7. Behavior(analysis) of sequences

Problem 7.1(104-Met.Rec)

Note that $(a_1 + a_3) + 2(a_2 + a_4) \le 2a_2 + 4a_3 \iff a_1 + 2a_4 \le 3a_3$ and $(a_1 + 2a_4) + 3(a_3 + a_5) \le 3a_3 + 6a_4 \iff a_1 + 3a_5 \le 4a_4$.
Let $n \ge 4$.Then for any natural $2 \le k \le n-2$, assuming that $a_1 + (k-1)a_{k+1} \le ka_k$ we obtain $(a_1 + (k-1)a_{k+1}) + k(a_k + a_{k+2}) \le ka_k + 2ka_{k+1} \iff a_1 + ka_{k+2} \le ka_{k+1}$.Thus, by Math Induction, we proved $a_1 + (k-1)a_{k+1} \le ka_k$ for any $k = 2, ..., n-1$.Since $a_1 = a_n = 0$ then

$$a_1 + (n-2)a_n \le na_{n-1} \implies 0 \le na_{n-1} \iff 0 \le a_{n-1}.$$

Since $0 \le a_{n-1}$ then

$$a_1 + (n-3)a_{n-1} \le (n-1)a_{n-2} \implies 0 \le (n-1)a_{n-2} \iff 0 \le a_{n-2}$$

Assuming $a_{n-i} \ge 0$ for any $1 \le i \le n-2$, since $a_1 + (n-i-2)a_{n-i} \le (n-i)a_{n-i-1}$, we obtain $0 \le (n-i)a_{n-i-1} \iff 0 \le a_{n-i-1}$.

Thus, by Math Induction $a_k \ge 0, i = 1, 2, ..., n$.

Problem 7.2(105-Met. Rec.)

a) Since $a_n \uparrow \mathbb{N}$ then $a_n > 1$, $n > 1$ and we obtain

$$a_{n+1}^2=\left(a_n+\frac{1}{a_n}\right)^2=a_n^2+\frac{1}{a_n^2}+2 \implies a_n^2+2<a_{n+1}^2<a_n^2+3 \implies$$

$$a_1^2+2(n-1)<a_n^2<a_1^2+3(n-1) \iff \sqrt{2n-1}<a_n<\sqrt{3n-2}, n>1$$

More precisely, from $\sqrt{2n-1}<a_n$ follows that

$$a_{n+1}=a_n+\frac{1}{a_n}<a_n+\frac{1}{\sqrt{2n-1}}$$

.

That imply

$$a_{n+1}-a_1<\sum_{k=1}^{n}\frac{1}{\sqrt{2n-1}} \iff a_{n+1}-2<\sum_{k=2}^{n}\frac{1}{\sqrt{2k-1}} \implies$$

$$a_{n+1}-2<\sum_{k=1}^{n-1}\frac{1}{\sqrt{2k+1}}<\sum_{k=1}^{n-1}\frac{2}{\sqrt{2k+1}+\sqrt{2k-1}}=\sum_{k=1}^{n-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)=$$

$$\sqrt{2n-1}-1 \implies a_{n+1}<\sqrt{2n-1}+1.$$

Thus, for any $n>1$ holds $\sqrt{2n-1}<a_n<\sqrt{2n-3}+1$.In particularly for $n=100$ we obtain $14<\sqrt{199}<a_{100}<\sqrt{197}+1<15$.Also from this inequality follows $\lim\limits_{n\to\infty}\dfrac{a_n}{\sqrt{n}}=\sqrt{2}$.

b)

i. Since $a_n\uparrow\mathbb{N}$ then $a_n>1$, $n>1$ and we obtain

$$a_{n+1}^3=\left(a_n+\frac{1}{a_n^2}\right)^3=a_n^3+3+\frac{3}{a_n^3}+\frac{1}{a_n^6}>a_n^3+3 \implies$$

$$a_n^3>a_1^3+3(n-1)=3n-2 \iff a_n>\sqrt[3]{3n-2}.$$

iii. $a_{n+1}=a_n+\dfrac{1}{a_n^2}<a_n+\dfrac{1}{\sqrt[3]{(3n-2)^2}} \implies a_{n+1}-2<\displaystyle\sum_{k=2}^{n}\frac{1}{\sqrt[3]{(3k-2)^2}} \implies$

$$a_{n+1}-2<\sum_{k=1}^{n-1}\frac{1}{\sqrt[3]{(3k+1)^2}}<\sum_{k=1}^{n-1}\frac{3}{\sqrt[3]{(3k+1)^2}+\sqrt[3]{(3k+1)(3k-2)}+\sqrt[3]{(3k-2)^2}}=$$

$$\sum_{k=1}^{n-1}\left(\sqrt[3]{3k+1}-\sqrt[3]{3k-2}\right)=\sqrt[3]{3n-2}-1 \implies a_{n+1}<\sqrt[3]{3n-2}+1.$$

Thus, for any $n>1$ holds $\sqrt[3]{3n-2}<a_n\le\sqrt[3]{3n-5}+1$
(for $n>2$ holds $\sqrt[3]{3n-2}<a_n<\sqrt[3]{3n-5}+1$) and $\lim\limits_{n\to\infty}\dfrac{a_n}{\sqrt[3]{n}}=\sqrt[3]{3}$.

ii. Lower bound $\sqrt[3]{3n-2}$ for a_n isn't good enough to provide proof of inequality $a_{9000}>30$. Starting from $a_2=2$ in inequality $a_{n+1}^3>a_n^3+3$ we obtain that $a_n^3>a_2^3+3(n-2)=8+3n-6=3n+2$ and this gives sharper lower bound $\sqrt[3]{3n+2}$ for a_n.Thus $a_{9000}>\sqrt[3]{27002}>30$.

From the other hand this lower bound gives us sharper upper bound for a_n :

$a_{n+1} = a_n + \dfrac{1}{a_n^2} < a_n + \dfrac{1}{\sqrt[3]{(3n+2)^2}} \implies$

$a_{n+1} - 1 < \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt[3]{(3k+2)^2}} < \sum\limits_{k=1}^{n} \dfrac{3}{\sqrt[3]{(3k+2)^2} + \sqrt[3]{(3k+2)(3k-1)} + \sqrt[3]{(3k-1)^2}} =$

$\sum\limits_{k=1}^{n} \left(\sqrt[3]{3k+2} - \sqrt[3]{3k-1}\right) = \sqrt[3]{3n+2} - \sqrt[3]{2} \implies a_{n+1} < \sqrt[3]{3n+2} + 1 - \sqrt[3]{2}.$

Or, $a_{n+1} - a_1 = \dfrac{1}{a_1^2} + \dfrac{1}{a_2^2} + \sum\limits_{k=3}^{n} \dfrac{1}{a_k^2} < 1 + \dfrac{1}{4} + \sum\limits_{k=3}^{n} \dfrac{1}{\sqrt[3]{(3k+2)^2}} <$

$$1 + \frac{1}{4} + \sum_{k=1}^{n} \left(\sqrt[3]{3k+2} - \sqrt[3]{3k-1}\right) = 1 + \frac{1}{4} + \sqrt[3]{3n+2} - \sqrt[3]{3\cdot 3 - 1} = \frac{1}{4} + \sqrt[3]{3n+2} - 1.$$

Thus, $a_{n+1} < \dfrac{1}{4} + \sqrt[3]{3n+2} \iff a_n < \dfrac{1}{4} + \sqrt[3]{3n-1}$ and finally we obtain

$$\sqrt[3]{3n+2} < a_n < \frac{1}{4} + \sqrt[3]{3n-1}.$$

Problem 7.3(106-Met. Rec.)

$a_{n+1} = 2^n - 3a_n \iff a_{n+1} = \dfrac{2^{n+1}}{5} + \dfrac{3 \cdot 2^n}{5} - 3a_n \iff$

$a_{n+1} - \dfrac{2^{n+1}}{5} = -3\left(a_n - \dfrac{2^n}{5}\right), n \in \mathbb{N} \cup \{0\} \implies$

$$a_n - \frac{2^n}{5} = (-3)^n \left(a_0 - \frac{2^0}{5}\right) = (-3)^n \left(a - \frac{1}{5}\right) \iff a_n = \frac{2^n}{5} + (-3)^n \left(a - \frac{1}{5}\right), n \in \mathbb{N} \cup \{0\}.$$

Now we claim

$$a_{n+1} > a_n \iff \frac{2^{n+1}}{5} + (-3)^{n+1}\left(a - \frac{1}{5}\right) > \frac{2^n}{5} + (-3)^n \left(a - \frac{1}{5}\right) \iff$$

$$\frac{2^n}{5} + 4(-3)^{n+1}\left(a - \frac{1}{5}\right) > 0, n \in \mathbb{N} \cup \{0\}$$

For $n = 2m$ we have

$\dfrac{2^{2m}}{5} + 4(-3)^{2m+1}\left(a - \dfrac{1}{5}\right) > 0 \iff \dfrac{2^{2m}}{5} > 12(-3)^{2m}\left(a - \dfrac{1}{5}\right) \iff$

$\dfrac{1}{60}\left(\dfrac{4}{9}\right)^m > a - \dfrac{1}{5}$

and for $n = 2m - 1$ we obtain

$$\frac{2^{2m-1}}{5} + 4(-3)^{2m}\left(a - \frac{1}{5}\right) > 0 \iff a - \frac{1}{5} > -\frac{1}{40}\left(\frac{4}{9}\right)^m.$$

Since $-\dfrac{1}{40}\left(\dfrac{4}{9}\right)^m < a-\dfrac{1}{5} < \dfrac{1}{60}\left(\dfrac{4}{9}\right)^m$ then

$$\lim_{m\to\infty}\left(-\frac{1}{40}\left(\frac{4}{9}\right)^m\right) \le a-\frac{1}{5} \le \lim_{m\to\infty}\frac{1}{60}\left(\frac{4}{9}\right)^m \iff 0\le a-\frac{1}{5}\le 0 \iff a=\frac{1}{5}.$$

Problem 7.4(107-Met. Rec.)

Assume that $a_1>0$ (because if $a_1\le 0$ then $n_0=2$).If $b=0$ then inequality $a_{n+1}\le\left(1+\dfrac{b}{n}\right)a_n-1, n\in\mathbb{N}$ becomes

$$a_{n+1}\le a_n-1, n\in\mathbb{N} \iff a_{n+1}+n+1\le a_n+n, n\in\mathbb{N}.$$

Hence, $a_n+n\le a_1+1 \iff a_n\le a_1+1-n, n\in\mathbb{N}$ and, therefore, for $n_0=[a_1]+2$ we obtain $a_{n_0}<0$.Let $b\in(0,1)$ let $\dfrac{p}{q}$ be fraction such that $b\le\dfrac{p}{q}<1$.Then

$$a_{n+1}\le\left(1+\frac{b}{n}\right)a_n-1, n\in\mathbb{N} \implies a_{n+1}\le\left(1+\frac{p}{nq}\right)a_n-1, n\in\mathbb{N} \iff$$

$$a_{n+1}\le\frac{nq+p}{nq}\cdot a_n-1, n\in\mathbb{N} \implies a_{n+1}\le\frac{nq+p}{(n-1)q+p}\cdot a_n-1, n\in\mathbb{N} \iff$$

$$\frac{a_{n+1}}{nq+p}\le\frac{a_n}{(n-1)q+p}-\frac{1}{nq+p}, n\in\mathbb{N}.$$

Hence, $\displaystyle\sum_{k=1}^{n}\left(\frac{a_{k+1}}{kq+p}-\frac{a_k}{(k-1)q+p}\right)\le-\sum_{k=1}^{n}\frac{1}{kq+p} \iff$

$$\frac{a_{n+1}}{nq+p}-\frac{a_1}{(1-1)q+p}\le-\sum_{k=1}^{n}\frac{1}{kq+p} \iff \frac{a_{n+1}}{nq+p}\le\frac{a_1}{p}-\sum_{k=1}^{n}\frac{1}{kq+p} \implies$$

$$\frac{a_{n+1}}{nq+p}\le\frac{a_1}{p}-\sum_{k=1}^{n}\frac{1}{(k+1)q} \iff \frac{a_{n+1}}{nq+p}\le\frac{a_1}{p}-\frac{1}{q}\sum_{k=1}^{n}\frac{1}{k+1} \iff$$

$$\frac{a_{n+1}}{nq+p}\le\frac{a_1}{p}+\frac{1}{q}-\frac{h_{n+1}}{q}, n\in\mathbb{N} \iff \frac{a_n}{(n-1)q+p}\le\frac{a_1}{p}+\frac{1}{q}-\frac{h_n}{q}, n\in\mathbb{N},$$

where $h_n=1+\dfrac{1}{2}+...+\dfrac{1}{n}$.

Since sequence $(h_n)_{n\in\mathbb{N}}$ have no upper bound (unbounded from above)*

that is for any $M>0$ there is $n\in\mathbb{N}$ such that $h_n>M$ then, in particular, for $M=\dfrac{a_1q}{p}+1$ there is n_0 such that $h_{n_0}>\dfrac{a_1q}{p}+1 \iff \dfrac{a_1}{p}+\dfrac{1}{q}-\dfrac{h_{n+1}}{q}$ and, therefore, $\dfrac{a_{n_0}}{(n_0-1)q+p}\le\dfrac{a_1}{p}+\dfrac{1}{q}-\dfrac{h_{n_0}}{q}<0 \implies a_{n_0}<0$.

*Noting that $h_{2^{n+1}}-h_{2^n}=\displaystyle\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k}>\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{2^{n+1}}=\frac{2^{n+1}-2^n}{2^{n+1}}=\frac{2^n}{2^{n+1}}=\frac{1}{2}$ we obtain $\displaystyle\sum_{k=1}^{n}\left(h_{2^k}-h_{2^{k-1}}\right)>\frac{n}{2} \iff h_{2^n}-h_{2^{1-1}}>\frac{n}{2} \iff$

$h_{2^n} - h_1 > \dfrac{n}{2} \iff h_{2^n} > \dfrac{n}{2} + 1$.Let M be any positive real number. Then for any natural $n > 2(M-1)$ we have $h_{2^n} > \dfrac{n}{2} + 1 > \dfrac{2(M-1)}{2} + 1 = M$.

Or, we can prove that h_n unbounded from above by another way, namely noting that

$$\left(1+\frac{1}{n}\right)^n < e \iff 1+\frac{1}{n} < e^{1/n} \iff \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \iff \ln(n+1) - \ln n < \frac{1}{n}$$

we obtain $h_n = \sum_{k=1}^{n} \dfrac{1}{k} > \sum_{k=1}^{n} (\ln(k+1) - \ln k) = \ln(n+1) - \ln 1 = \ln(n+1)$.

Problem 7.5*(109-Met.Rec.) (Team Selection Test, Singapur)

Let $n \in \mathbb{N}$, $a_0 = \dfrac{1}{2}$ and $a_{k+1} = a_k + \dfrac{a_k^2}{n}, k \in \mathbb{N}$. Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

Solution.

$a_{k+1} = a_k + \dfrac{a_k^2}{n} \iff \dfrac{a_{k+1}}{n} = \dfrac{a_k}{n} + \dfrac{a_k^2}{n^2} \iff b_{k+1} = b_k + b_k^2$, where $b_k := \dfrac{a_k}{n}$.Since $b_0 = \dfrac{1}{2n}$ we have equivalent problem:

Let $n \in \mathbb{N}$, $b_0 = \dfrac{1}{2n}$ and $b_{k+1} = b_k + b_k^2, k \in \mathbb{N}\cup\{0\}$. Prove that

$$\frac{1}{n} - \frac{1}{n^2} < a_n < \frac{1}{n}.$$

Since, $\dfrac{1}{b_{k+1}} = \dfrac{1}{b_k + b_k^2} = \dfrac{1}{b_k} - \dfrac{1}{b_k+1} \iff \dfrac{1}{b_k+1} = \dfrac{1}{b_k} - \dfrac{1}{b_{k+1}}$ we have

$$\sum_{k=0}^{n-1} \frac{1}{b_k+1} = \frac{1}{b_0} - \frac{1}{b_n} = 2n - \frac{1}{b_n}.$$

Note that $b_k \uparrow \mathbb{N}$, because $b_{k+1} = b_k + b_k^2 > b_k, k \in \mathbb{N}\cup\{0\}$.

Hence $b_k \geq b_0 = \dfrac{1}{2n}$ and

$$2n - \frac{1}{b_n} \leq \sum_{k=0}^{n-1} \frac{1}{b_0+1} = \frac{n}{b_0+1} = \frac{n}{\frac{1}{2n}+1} = \frac{2n^2}{2n+1} \iff$$

$$\frac{1}{b_n} \geq 2n - \frac{2n^2}{2n+1} = \frac{2n(n+1)}{2n+1} \iff b_n \leq \frac{2n+1}{2n(n+1)}.$$

Since $\dfrac{2n+1}{2n(n+1)} < \dfrac{1}{n} \iff 2n+1 < 2n+2$ we obtain $b_n < \dfrac{1}{n}$.

From $b_n < \dfrac{1}{n}$ and $b_k \uparrow \mathbb{N}$ follows that for all $0 \leq k < n$ holds $b_k < \dfrac{1}{n}$.

Using this we obtain $2n - \dfrac{1}{b_n} = \sum_{k=0}^{n-1} \dfrac{1}{b_k+1} > \sum_{k=0}^{n-1} \dfrac{1}{\frac{1}{n}+1} = \dfrac{n^2}{n+1} \iff$

$$\frac{1}{b_n} < 2n - \frac{n^2}{n+1} = \frac{n(n+2)}{n+1} \iff b_n > \frac{n+1}{n(n+2)} \text{ and since}$$

$$\frac{n+1}{n(n+2)} > \frac{1}{n} - \frac{1}{n^2} \iff \frac{n+1}{n+2} > \frac{n-1}{n} \iff n^2+n > n^2+2n-n-2 \iff 2>0,$$

we finally get $b_n > \dfrac{1}{n} - \dfrac{1}{n^2}$.

Problem 7.6(110-Met. Rec.)
Let $a_n := \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ then

$$a_0 = 2, a_1 = 4 \text{ and } a_{n+1} - 4a_n + a_{n-1} = 0, n \in \mathbb{N}.$$

Since a_n is integer for any $n \in \mathbb{N} \cup \{0\}$ and
$\left(2+\sqrt{3}\right)^n = a_n - \left(2-\sqrt{3}\right)^n = a_n - 1 + 1 - \left(2-\sqrt{3}\right)^n = a_n - 1 + 1 - \dfrac{1}{\left(2+\sqrt{3}\right)^n}$
then $\left[\left(2+\sqrt{3}\right)^n\right] = a_n - 1$ *and* $\left\{\left(2+\sqrt{3}\right)^n\right\} = 1 - \dfrac{1}{\left(2+\sqrt{3}\right)^n}$.
Hence, $\lim\limits_{n\to\infty} \left\{\left(2+\sqrt{3}\right)^n\right\} = \lim\limits_{n\to\infty} \left(1 - \dfrac{1}{\left(2+\sqrt{3}\right)^n}\right) = 1.$

Problem 7.7 (111-Met. Rec.)
a) From the recurrence $x_{n+1} = x_n(1-x_n)$, $n \in \mathbb{N}\cup\{0\}$ and $x_0 \in (0,1)$ immediatelly follows that x_n is positive for any $n \in \mathbb{N}\cup\{0\}$. Since $x_{n+1} - x_n = -x_n^2$ then $(x_n)_{n\geq 0}$ is decreasing sequence. Thus (x_n) converge to some number a and since $x_{n+1} = x_n(1-x_n) \leq \dfrac{1}{4}$ then $a \in \left[0, \dfrac{1}{4}\right]$ and $a = a(1-a)$.
Therefore $\lim\limits_{n\to\infty} x_n = 0$.
For any $k \in \mathbb{N}\cup\{0\}$ we have $\dfrac{1}{x_{k+1}} = \dfrac{1}{x_k(1-x_k)} = \dfrac{1}{x_k} + \dfrac{1}{1-x_k} \iff$
$\dfrac{1}{x_{k+1}} - \dfrac{1}{x_k} = \dfrac{1}{1-x_k}$ and since $\dfrac{1}{1-x_k} > 1$, then
$\dfrac{1}{x_n} - \dfrac{1}{x_0} = \sum\limits_{k=0}^{n-1} \left(\dfrac{1}{x_{k+1}} - \dfrac{1}{x_k}\right) = \sum\limits_{k=0}^{n-1} \dfrac{1}{1-x_k} > n$ for any $n \in \mathbb{N}$.
Thus, we have inequality
(1) $\dfrac{1}{x_n} > \dfrac{1}{x_0} + n$, which can be rewritten in the form
(2) $\dfrac{1}{nx_n} > 1 + \dfrac{1}{nx_0}$
and in the form $x_n < \dfrac{x_0}{1+nx_0} < \dfrac{1}{n}$.
Since $x_n < \dfrac{1}{n}, n \in \mathbb{N}$ and $\dfrac{1}{x_n} - \dfrac{1}{x_2} = \sum\limits_{k=2}^{n-1} \dfrac{1}{1-x_k}$ then

$$\frac{1}{x_n} - \frac{1}{x_2} < \sum_{k=2}^{n-1} \frac{1}{1-\frac{1}{k}} = \sum_{k=2}^{n-1} \frac{k}{k-1} = \sum_{k=2}^{n-1} \left(1 + \frac{1}{k-1}\right) = n-2+h_{n-2} < n+h_n,$$

where $h_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$.So, we have inequality

(3) $\quad 1 + \frac{1}{nx_0} < \frac{1}{nx_n} < 1 + \frac{1}{nx_2} + \frac{h_n}{n}$.

Since $\frac{h_n}{n}$ decreasing on $\mathbb{N}$ ($\frac{h_n}{n} > \frac{h_{n+1}}{n+1} \iff nh_n + h_n > nh_n + \frac{n}{n+1} \iff h_n > \frac{n}{n+1} \Longleftarrow h_n > 1$) and $\frac{h_{n^2}}{n^2} < \frac{2}{n}$

($\frac{h_{n^2}}{n^2} = \frac{h_n}{n^2} + \frac{h_{n^2} - h_n}{n^2} < \frac{1 \cdot n}{n^2} + \frac{(n^2 - n) \cdot \frac{1}{n+1}}{n^2} < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$)

then $\lim_{n\to\infty} \frac{h_n}{n} = 0$.Therefore $\lim_{n\to\infty} \left(1 + \frac{1}{nx_2} + \frac{h_n}{n}\right) = 1$ and since,

$\lim_{n\to\infty} \left(1 + \frac{1}{nx_0}\right) = 1$ as well, then $\lim_{n\to\infty} \frac{1}{nx_n} = 1 \iff \lim_{n\to\infty} nx_n = 1$.

Or, alternatively, $\lim_{n\to\infty} \frac{h_n}{n} = 0$ because

$$\frac{h_n}{n} < \left(\frac{1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}}{n}\right)^{\frac{1}{2}} < \left(\frac{2}{n}\right)^{\frac{1}{2}}.$$

Comment .

Using Arithmetic Mean Limit Theorem, we can easy prove that $\lim_{n\to\infty} \frac{h_n}{n} = 0$.

(AML Theorem: If $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = a$).

And for those, who are familiar with Shtolz Theorem, this problem became simple exercise on it's application, namely

$$\lim_{n\to\infty} \frac{1}{nx_n} = \lim_{n\to\infty} \frac{\frac{1}{x_n}}{n} = \lim_{n\to\infty} \frac{\frac{1}{x_n} - \frac{1}{x_{n-1}}}{n - (n-1)} = \lim_{n\to\infty} \frac{\frac{1}{1 - x_{n-1}}}{1} = \frac{1}{1 - \lim_{n\to\infty} x_{n-1}} = 1.$$

bi) First note that $x_n > 1$ for all $n \in \mathbb{N}$.Really, $x_1 = a > 1$ and from supposition $x_n > 1$ follows $x_{n+1} = x_n(x_n - 1) + 1 > 1$.Moreover, $x_{n+1} - x_n = (x_n - 1)^2 > (a-1)^2$

and this imply $x_{n+1} > a + (n-1)(a-1)^2$.

Since $x_{n+1} = x_n^2 - x_n + 1 \iff x_{n+1} - 1 = x_n(x_n - 1) \iff$

$$\frac{1}{x_{n+1} - 1} = \frac{1}{x_n(x_n - 1)} = \frac{1}{x_n - 1} - \frac{1}{x_n} \iff \frac{1}{x_n - 1} - \frac{1}{x_{n+1} - 1} = \frac{1}{x_n}$$

we have

$$\sum_{k=1}^{n} \frac{1}{x_k} = \sum_{k=1}^{n} \left(\frac{1}{x_k - 1} - \frac{1}{x_{k+1} - 1}\right) = \frac{1}{a-1} - \frac{1}{x_{n+1} - 1}$$

and

$$0 < \frac{1}{a-1} - \sum_{k=1}^{n} \frac{1}{x_k} = \frac{1}{x_{n+1}-1} < \frac{1}{a+(n-1)(a-1)^2}.$$

So, $\sum\limits_{n=1}^{\infty} \frac{1}{x_n} = \frac{1}{a-1}$.

bii) Since x

$$x_{n+1} = x_n^2 - x_n + 1 \iff x_{n+1} - 1 = x_n (x_n - 1) \iff x_n = \frac{x_{n+1}-1}{x_n - 1}$$

then

$$x_1 x_2 ... x_n = \frac{x_{n+1}-1}{x_1 - 1} = \frac{x_{n+1}-1}{a-1} \text{ and } \frac{x_{n+1}}{x_1 x_2 ... x_n} = \frac{x_{n+1}(a-1)}{x_{n+1}-1} > a-1.$$

From the other hand, since $\frac{x_{n+1}}{x_n} = x_n - \left(1 - \frac{1}{x_n}\right) < x_n$ then $\frac{x_{n+1}}{x_n^2} < 1,$ and denoting $p_n := \frac{x_{n+1}}{x_1 x_2 ... x_n},$ we obtain

$$p_n = p_{n-1} \cdot \frac{x_{n+1}}{x_n^2} < p_{n-1}, n > 1.$$

Therefore, $p_n < p_1 = x_1 = a$ and $a - 1 < \frac{x_{n+1}}{x_1 x_2 ... x_n} < a$.

If a is integer then $\left\lfloor \frac{x_{n+1}}{x_1 x_2 ... x_n} \right\rfloor = a - 1$.

Remark.

Since $\lim\limits_{n\to\infty} x_n = \infty$ then $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_1 x_2 ... x_n} = \lim\limits_{n\to\infty} \frac{x_{n+1}(a-1)}{x_{n+1}-1} = a - 1$.

c) For convenience we will use substitution $x_n = -a_n, n \in \mathbb{N} \cup \{0\}$.

Then $a_0 = -\frac{1}{3}$ and $a_{n+1} = 1 - \frac{1}{2} a_n^2, n \in \mathbb{N} \cup \{0\}$.

Note that $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Really, $a_1 = \frac{17}{18}$ and from supposition $a_n \in (0,1)$ immediatelly follows

$$a_{n+1} = 1 - \frac{1}{2} a_n^2 \in \left(\frac{1}{2}, 1\right) \subset (0,1).$$

Suppose now that $a = \lim\limits_{n\to\infty} a_n$ then $a = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(1 - \frac{1}{2} a_n^2\right) = 1 - \frac{1}{2} \lim\limits_{n\to\infty} a_n^2 = 1 - \frac{1}{2} a^2$. Since equation $a = 1 - \frac{1}{2} a^2$ has only one solution in $(0,1)$,namely, $a = \sqrt{3} - 1$.Then only a_* can be the limit of sequence $(a_n)_{n\geq 1}$. So, suffice to prove that sequence $(a_n)_{n\geq 1}$ converge to a.

For any $n \in \mathbb{N}$ we have, $|a_{n+1} - a| = \left|1 - \frac{1}{2} a_n^2 - 1 - \frac{1}{2} a^2\right| = \frac{1}{2} \left|a_n^2 - a^2\right| =$

$$\frac{1}{2}|a_n-a|\cdot(a_n+a)<\frac{1}{2}|a_n-a|\cdot\left(1+\sqrt{3}-1\right)=\frac{\sqrt{3}}{2}|a_n-a|.$$

Hence, $|a_n-a|<\left(\dfrac{\sqrt{3}}{2}\right)^{n-1}|a_1-a|$ and that imply $\lim\limits_{n\to\infty}|a_n-a|=0 \iff \lim\limits_{n\to\infty}a_n=a$.

Problem 7.8 (112-Met. Rec.)

Note that $x_{n+1}=0.5x_n^2-1 \iff \dfrac{x_{n+1}}{2}=\dfrac{x_n^2}{4}-\dfrac{1}{2} \iff$
$-\dfrac{x_{n+1}}{2}=\dfrac{1}{2}-\left(-\dfrac{x_n}{2}\right)^2 \iff a_{n+1}=\dfrac{1}{2}-a_n^2$,
where $a_n:=-\dfrac{x_n}{2}, n\in\mathbb{N}\cup\{0\}$ and $a_0=-1/6$.
We will prove that $a_n\in(0,1/2), n\geq 1$.
Indeed, $a_1=\dfrac{1}{2}-a_0^2=\dfrac{1}{2}-\dfrac{1}{36}=\dfrac{17}{36}\in(0,1/2)$ and for any $n\in\mathbb{N}$ supposition $a_n\in(0,1/2)$ yields $-1/4<-a_n^2<0 \implies$
$\dfrac{1}{4}<\dfrac{1}{2}-a_n^2<\dfrac{1}{2} \iff a_{n+1}\in(0,1/2)$.
So, by Math Induction $a_n\in(0,1/2), n\in\mathbb{N}$.
Suppose that sequence $(a_n)_{n\geq 0}$ converge and $a:=\lim\limits_{n\to\infty}a_n$ then $a\geq 0$ and
$a=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\left(\dfrac{1}{2}-a_n^2\right)=\dfrac{1}{2}-\lim\limits_{n\to\infty}a_n^2=\dfrac{1}{2}-a^2 \implies$
$a^2+a-\dfrac{1}{2}=0 \overset{a\geq 0}{\iff} a=\dfrac{-1+\sqrt{3}}{2}$
We will prove that $\lim\limits_{n\to\infty}a_n=a$.
Since $0<a_n+a<\dfrac{\sqrt{3}-1}{2}+\dfrac{1}{2}=\dfrac{\sqrt{3}}{2}$ then for any $n\in\mathbb{N}$ we have

$$|a_{n+1}-a|=\left|\frac{1}{2}-a_n^2-a\right|=\left|\frac{1}{2}-a_n^2-\left(\frac{1}{2}-a^2\right)\right|=$$

$$\left|a_n^2-a^2\right|=|a_n-a|\cdot|a_n+a|<\frac{\sqrt{3}}{2}|a_n-a|.$$

Hence, $|a_n-a|<\left(\dfrac{\sqrt{3}}{2}\right)^{n-1}|a_1-a|$.

Since $a_n=-\dfrac{x_n}{2}$ then $\lim\limits_{n\to\infty}x_n=-2\lim\limits_{n\to\infty}a_n=(-2)\cdot\dfrac{\sqrt{3}-1}{2}=1-\sqrt{3}$.

Problem 7.9*

a) $a_1=\dfrac{1}{2}, a_2=\dfrac{1}{2}-\dfrac{1}{4}=\dfrac{1}{4}, a_3=\dfrac{1}{4}-\dfrac{2}{16}=\dfrac{1}{4}-\dfrac{1}{8}=\dfrac{1}{8}$.

1. $a_{n+1}=a_n-na_n^2=\dfrac{na_n(1-na_n)}{n}\leq\dfrac{1}{4n} \implies$

$$a_n \le \frac{1}{4(n-1)}, n > 1 \implies a_n \le \frac{1}{2n}, n \in \mathbb{N}.$$

2. Since $a_n \le \dfrac{1}{2n}, n \in \mathbb{N}$ and $a_{n+1} = a_n - na_n^2 \iff \dfrac{1}{a_{n+1}} = \dfrac{1}{a_n} + \dfrac{n}{1 - na_n}$ we obtain $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} \le \dfrac{n}{1 - \frac{n}{2n}} = 2n$ and

$$\frac{1}{a_{n+1}} - \frac{1}{a_1} = \sum_{k=1}^{n} \frac{k}{1 - ka_k} \le \sum_{k=1}^{n} 2n = n(n+1) \implies$$

$$\frac{1}{a_{n+1}} - 2 \le n(n+1) \iff a_n \ge \frac{1}{2 + n(n-1)} \ge \frac{1}{n(n+1)}.$$

So, $\dfrac{1}{n(n+1)} \le a_n \le \dfrac{1}{2n}$.

But obtained upper bound for a_n is not good enough to prove inequality

$$a_1 + a_2 + ... + a_n < \frac{3}{2}.$$

Then, using $\dfrac{1}{n(n+1)} \le a_n$ and $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_1} = \sum\limits_{k=1}^{n} \dfrac{k}{1 - ka_k}$, we get

$$\frac{1}{a_{n+1}} - 2 = \sum_{k=1}^{n} \frac{k}{1 - ka_k} \ge \sum_{k=1}^{n} \frac{k}{1 - k \cdot \frac{1}{k(k+1)}} = \sum_{k=1}^{n} (k+1) = \frac{(n+1)(n+2)}{2} - 1 \iff$$

$$\frac{1}{a_{n+1}} \ge \frac{(n+1)(n+2)}{2} + 1 \implies a_n \le \frac{2}{n(n+1)}.$$

Hence, $a_1 + a_2 + ... + a_n \le 2\left(1 - \dfrac{1}{n+1}\right) < 2.$

But we need $a_1 + a_2 + ... + a_n < \dfrac{3}{2}$.To get this upper bound we will use in the sum $a_1 + a_2 + ... + a_n$ new estimation of a$_n$, starting from $n \ge 4$.

So, for $n \ge 4$ we have

$$a_1 + a_2 + ... + a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \sum_{k=4}^{n} a_k < \frac{7}{8} + 2\sum_{k=4}^{n} \frac{1}{k(k+1)} =$$

$$\frac{7}{8} + 2\left(\frac{1}{4} - \frac{1}{n+1}\right) < \frac{7}{8} + \frac{1}{2} = \frac{11}{8} < \frac{3}{2}.$$

Since $a_1, a_1 + a_2, a_1 + a_2 + a_3 < \dfrac{3}{2}$ then inequality

$a_1 + a_2 + ... + a_n < \dfrac{3}{2}$ proved for all $n \in \mathbb{N}$.

Second solution.

Since $a_{n+1} - a_n = -na_n^2 < 0$ then a_n decreasing sequence and since

$$a_{n+1} = a_n - na_n^2 = \frac{na_n(1 - na_n)}{n} \le \frac{1}{4n} < \frac{1}{n+1}$$

then $0 < a_n < \dfrac{1}{n}$ for all $n \in \mathbb{N}$.
From $0 < a_n < \dfrac{1}{n}$ and $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} = \dfrac{n}{1 - na_n}$ follows
$\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} > n$ for all $n \in \mathbb{N}$ and then

$$\frac{1}{a_{n+1}} - \frac{1}{a_1} = \sum_{k=1}^{n}\left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right) > \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \implies$$

$$\frac{1}{a_{n+1}} > 2 + \frac{n(n+1)}{2} = \frac{n^2+n+4}{2} \implies$$

$$a_{n+1} < \frac{2}{n^2+n+4} < \frac{2}{n(n+1)}, n \in \mathbb{N} \implies$$

$$a_n \le \frac{2}{(n-1)n}, n \ge 2$$

(since $a_1 = \dfrac{1}{2} = \dfrac{2}{1^2 - 1 + 4}$ then $a_n < \dfrac{2}{n^2 - n + 4}, n \in \mathbb{N}$).
Further the same as above.
Remark.
Little bit worse upper bound can be obtained using AM-GM inequality, namely, using first estimation $a_{n+1} \le \dfrac{1}{n+1}$, equalities
$1 - na_n = \dfrac{a_{n+1}}{a_n}$ and $\dfrac{1}{a_{n+1}} - 2 = \sum\limits_{k=1}^{n} \dfrac{k}{1 - ka_k}$ we have

$$\frac{1}{a_{n+1}} - 2 = \sum_{k=1}^{n} \frac{ka_n}{a_{n+1}} \ge n\sqrt[n]{\frac{n!a_1}{a_{n+1}}} \ge n\sqrt[n]{(n+1)!} > \frac{n(n+1)}{3},$$

because $n! > \left(\dfrac{n}{3}\right)^n$.
That gives us $a_{n+1} < \dfrac{3}{n(n+1)} \implies a_n \le \dfrac{3}{n(n-1)}$ for $n > 1$.
b) Although the lower and upper bounds for a_n represented by double inequality

$$\frac{1}{n(n+1)} \le a_n \le \frac{2}{n(n+1)}$$

provide proof of inequality $a_1, a_1 + a_2, a_1 + a_2 + a_3 < \dfrac{3}{2}$,

they are still not good enough because $\lim\limits_{n\to\infty} \dfrac{\dfrac{2}{n(n+1)}}{\dfrac{1}{n(n+1)}} = 2 \ne 1$.
Thus, we somehow have to improve obtained bounds.
Since the function $h(x) := \dfrac{k}{1 - kx}$ increasing in $\left(0, \dfrac{1}{k}\right)$
and $\left(\dfrac{1}{k(k+1)}, \dfrac{2}{k(k+1)}\right) \subset \left(0, \dfrac{1}{k}\right)$ then

$$\frac{k}{1 - k \cdot \dfrac{1}{k(k+1)}} < \frac{k}{1 - ka_k} < \frac{k}{1 - k \cdot \dfrac{2}{k(k+1)}} \iff$$

$k+1 < \dfrac{k}{1-ka_k} < \dfrac{k(k+1)}{k-1}$.

Unfortunately, upper bound $\dfrac{k(k+1)}{k-1}$ is not convenient for further summation. But we can take $\dfrac{3}{n(n+3)}$, for any $n \geq 3$, as upper bound for a_n, instead $\dfrac{2}{n(n+1)}$ ($\dfrac{2}{n(n+1)} \leq \dfrac{3}{n(n+3)} \iff n \geq 3$). Then, since $\dfrac{3}{k(k+3)} < \dfrac{1}{k}$,we obtain $\dfrac{k}{1-ka_k} < \dfrac{k}{1-k\cdot\dfrac{3}{k(k+3)}} = k+3$.

Since $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_3} = \sum\limits_{k=3}^{n} \dfrac{k}{1-ka_k}$,then using inequality

$k+1 < \dfrac{k}{1-ka_k} < k+3, k=3,4,...,$ we obtain

$\sum\limits_{k=3}^{n}(k+1) < \dfrac{1}{a_{n+1}} - \dfrac{1}{a_3} < \sum\limits_{k=3}^{n}(k+3) \iff$

$\dfrac{(n+1)(n+2)}{2} - 6 + \dfrac{1}{a_3} < \dfrac{1}{a_{n+1}} < \dfrac{(n+3)(n+4)}{2} - 15 + \dfrac{1}{a_3} \iff$

$\dfrac{(n+1)(n+2)}{2} + 2 < \dfrac{1}{a_{n+1}} < \dfrac{(n+3)(n+4)}{2} - 7 \iff$

$\dfrac{(n+1)(n+2)}{2} < \dfrac{1}{a_{n+1}} < \dfrac{(n+3)(n+4)}{2} \implies$

$$\dfrac{2}{n(n+1)} < a_n < \dfrac{2}{(n+2)(n+3)}.$$

So, we get "good" bounds for a_n because

$$\lim_{n\to\infty} \dfrac{a_n}{\dfrac{2}{n(n+1)}} = \lim_{n\to\infty} \dfrac{\dfrac{2}{(n+2)(n+3)}}{\dfrac{2}{n(n+1)}} = 1,$$

i.e. we get asymptotic representation for a_n : $\quad a_n \sim \dfrac{2}{n(n+1)}$.

c) Follows immediately from **(b)**.

Problem 7.10

i. We will prove that $|a_n| \leq 1$. Really, for any $n \in \mathbb{N}$ suppose that $|a_n| \leq 1$. This imply $|a_n| \leq 2 \iff 0 \leq a_n^2 \leq 4 \iff -2 \leq a_n^2 - 2 \leq 2 \iff$
$\left|a_n^2 - 2\right| \leq 2 \iff |a_{n+1}| \leq 1$.

If $a_1 = 2$, then $a_2 = 1$ and result remains the same, namely $|a_n| \leq 1$ for all $n > 1$.

ii.

If $a_1 = 3$ *then the same recurrence defines an unbounded sequence.*

Really, if $a_1 = 3$ then $a_n \geq 3$ for all $n \in \mathbb{N}$ because from supposition $a_n \geq 3$ follows $a_{n+1} = \dfrac{a_n^2 - 2}{2} \geq \dfrac{9-2}{2} = 3.5 > 3$.Hereof

$a_{n+1} = \dfrac{a_n^2 - 2}{2} \geq \dfrac{3a_n - 2}{2} = \dfrac{3a_n}{2} - 1 \iff a_{n+1} - 2 \geq \dfrac{3(a_n - 1)}{2} \implies$
$a_n - 2 \geq \left(\dfrac{3}{2}\right)^{n-1}(a_1 - 2) = \left(\dfrac{3}{2}\right)^{n-1} \iff a_n \geq 2 + \left(\dfrac{3}{2}\right)^{n-1} \geq 3 + \dfrac{n-1}{2}.$

Problem 7.11

i. Since $a_{n+1} = \dfrac{3}{4}a_n + \dfrac{1}{a_n} \geq 2\sqrt{\dfrac{3}{4}a_n \cdot \dfrac{1}{a_n}} = \sqrt{3}$ then
$a_n \geq \sqrt{3}$ for all $n > 1$.
Assuming that M is upper bound for $(a_n)_{n\in\mathbb{N}}$, since $\sqrt{3} \leq a_n \leq M$ we obtain
$a_{n+1} = \dfrac{3}{4}a_n + \dfrac{1}{a_n} \leq \dfrac{3}{4}M + \dfrac{1}{\sqrt{3}}$
and we claim $\dfrac{3}{4}M + \dfrac{1}{\sqrt{3}} = M \iff M = \dfrac{4}{\sqrt{3}}$.
Note that $a_2 = \dfrac{7}{4} \in \left(\sqrt{3}, \dfrac{4}{\sqrt{3}}\right)$. Then by Math Induction we obtain

$$\sqrt{3} \leq a_n \leq \frac{4}{\sqrt{3}}, n \geq 2$$

ii. Noting that $2 = \dfrac{3}{4} \cdot 2 + \dfrac{1}{2}$ we obtain
$|a_{n+1} - 2| = \left|\dfrac{3}{4}a_n + \dfrac{1}{a_n} - \left(\dfrac{3}{4} \cdot 2 + \dfrac{1}{2}\right)\right| =$
$\left|\dfrac{3}{4}(a_n - 2) + \dfrac{2 - a_n}{2a_n}\right| = |a_n - 2|\left|\dfrac{3}{4} - \dfrac{1}{2a_n}\right|$.
Since $\sqrt{3} \leq a_n \leq \dfrac{4}{\sqrt{3}} \iff$
$\dfrac{1}{2\cdot(4/\sqrt{3})} \leq \dfrac{1}{2a_n} \leq \dfrac{1}{2\sqrt{3}} \iff \dfrac{\sqrt{3}}{8} \leq \dfrac{1}{2a_n} \leq \dfrac{\sqrt{3}}{6} \iff$
$\dfrac{3}{4} - \dfrac{\sqrt{3}}{6} \leq \dfrac{3}{4} - \dfrac{1}{2a_n} \leq \dfrac{3}{4} - \dfrac{\sqrt{3}}{8}$ and
$0 < \dfrac{3}{4} - \dfrac{\sqrt{3}}{6} < \dfrac{3}{4} - \dfrac{\sqrt{3}}{8} < \dfrac{2}{3}$
then $\left|\dfrac{3}{4} - \dfrac{1}{2a_n}\right| = \dfrac{3}{4} - \dfrac{1}{2a_n} < \dfrac{3}{4} - \dfrac{\sqrt{3}}{8} < \dfrac{2}{3}$.
Hence, $|a_{n+1} - 2| < \dfrac{2}{3}|a_n - 2|, n \geq 2 \implies |a_n - 2| < \left(\dfrac{2}{3}\right)^{n-2}|a_2 - 2| =$
$\left(\dfrac{2}{3}\right)^{n-2}\left|\dfrac{7}{4} - 2\right| = \left(\dfrac{2}{3}\right)^{n-2}\dfrac{1}{4} < \left(\dfrac{2}{3}\right)^n$.

Problem 7.12

By substitution $a_n := \dfrac{b_{n+1}}{b_n}$ in the recurrence $a_{n+1} = 1 + \dfrac{1}{a_n}$ we obtain

$\dfrac{b_{n+2}}{b_{n+1}} = 1 + \dfrac{b_n}{b_{n+1}} \iff b_{n+2} = b_{n+1} + b_n$.Since $a_1 = \dfrac{b_2}{b_1} = 1$ and $a_2 = \dfrac{b_3}{b_2} = 2$ we set $b_1 = 1, b_2 = 1$. Thus $b_n = f_n$ for $n \in \mathbb{N}$.
Since $f_{n+2} = f_{n+1} + f_n = 2f_n + f_{n-1}$ and $f_n = f_{n-1} + f_{n-2}$ we obtain $f_{n+2} - f_n = 2f_n + f_{n-1} - f_{n-1} - f_{n-2} = 2f_n - f_{n-2} \iff f_{n+2} = 3f_n - f_{n-2}$.
So, $f_{2n+3} = 3f_{2n+1} - f_{2n-1}$and $f_{2n+2} = 3f_{2n} - f_{2n-2}$.
We will prove:

i. $a_{2n-1} = \dfrac{f_{2n}}{f_{2n-1}} < \dfrac{f_{2n+1}}{f_{2n}} = a_{2n}$;

ii. $a_{2n-1} = \dfrac{f_{2n}}{f_{2n-1}} < \dfrac{f_{2n+2}}{f_{2n+1}} = a_{2n+1}$;

iii. $a_{2n} = \dfrac{f_{2n+1}}{f_{2n}} > \dfrac{f_{2n+3}}{f_{2n+2}} = a_{2n+2}$;

.

Proof.

i. $a_{2n-1} = \dfrac{f_{2n}}{f_{2n-1}} < \dfrac{f_{2n+1}}{f_{2n}} = a_{2n} \iff$

$f_{2n}^2 < f_{2n+1}f_{2n-1} \Longleftarrow f_{2n+1}f_{2n-1} - f_{2n}^2 = 1$
(follows from Cassini's identity $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$).
Moreover, from $0 \le a_{2n} - a_{2n-1} = \dfrac{1}{f_{2n-1}f_{2n}} < \dfrac{1}{f_{2n-1}^2}$ and

$\lim\limits_{n\to\infty} \dfrac{1}{f_{2n-1}^2} = 0$ follows
$\lim\limits_{n\to\infty} (a_{2n} - a_{2n-1}) = 0$.

ii,iii. By the same identity we obtain

$$f_{n+3}f_n - f_{n+1}f_{n+2} = (f_{n+2} + f_{n+1})\,f_n - (f_{n+1} + f_n)\,f_{n+1} = f_{n+2}f_n - f_{n+1}^2 = (-1)^{n+1}.$$

Thus, $a_{2n-1} < a_{2n+1} \iff$

$$\frac{f_{2n}}{f_{2n-1}} < \frac{f_{2n+2}}{f_{2n+1}} \iff f_{2n+2}f_{2n-1} - f_{2n}f_{2n+1} > 0 \Longleftarrow f_{2n+2}f_{2n-1} - f_{2n}f_{2n+1} = (-1)^{2n-1+1} = 1$$

and

$$a_{2n} > a_{2n+2} \iff \frac{f_{2n+1}}{f_{2n}} > \frac{f_{2n+3}}{f_{2n+2}} \iff f_{2n+3}f_{2n} - f_{2n+2}f_{2n+1} < 0 \Longleftarrow$$

$$f_{2n+3}f_{2n} - f_{2n+2}f_{2n+1} = (-1)^{2n+1} = -1.$$

So, $b := \lim\limits_{n\to\infty} a_{2n-1} = \lim\limits_{n\to\infty} a_{2n}$ and $a_{2n-1} < b < a_{2n}$.
Since $a_{2n} = 1 + \dfrac{1}{a_{2n-1}}$ then we obtain that b is positive root of equation $b = 1 + \dfrac{1}{b} \iff b^2 - b - 1 = 0$ that is $b = \dfrac{1+\sqrt{5}}{2}$.
Alternatively, we can directly, using Math. Induction prove that

(1) $\quad a_{2n-1} < \dfrac{1+\sqrt{5}}{2} < a_{2n}$.
Indeed, for any $n \in \mathbb{N}$ assuming **(1)** we obtain:
$a_{2n+1} = 1 + \dfrac{1}{a_{2n}} < 1 + \dfrac{2}{1+\sqrt{5}} = 1 + \dfrac{\sqrt{5}-1}{2} = \dfrac{1+\sqrt{5}}{2}$ and
$a_{2n+2} = 1 + \dfrac{1}{a_{2n+1}} > 1 + \dfrac{2}{1+\sqrt{5}} = \dfrac{1+\sqrt{5}}{2}$.
(Base of Math Induction is inequality $a_1 = \dfrac{f_2}{f_1} < \dfrac{1+\sqrt{5}}{2} < \dfrac{f_3}{f_2} = a_2$
wich obviously holds)
Another solution can by obtained from Binet formula for f_n and calculation of limits.

Problem 7.13

Since $a_n > 0$ for all n we will consider equivalent recurrence

$$a_{n+1}^2 = \frac{a_n^2}{4} + 1 + \frac{1}{a_n^2} \iff a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n},$$

which by substitution $a_n = \sqrt{\dfrac{p_n}{q_n}}$ can be rewritten in the form

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{4q_n} + 1 + \frac{q_n}{p_n} = \frac{(2q_n + p_n)^2}{4p_nq_n}.$$

Thus we can consider two recurrences
$p_{n+1} = (2q_n + p_n)^2$, $q_{n+1} = 4p_nq_n$, $n = 1, 2, ...,$ where $p_1 = 9, q_1 = 4$, since $a_1 = \dfrac{3}{2}$.
Let $b_n := \dfrac{2}{\sqrt{a_n^2 - 2}}$, then $b_n^2 = \dfrac{4}{a_n^2 - 2} = \dfrac{4q_n}{p_n - 2q_n}$ is integer because $p_n - 2q_n = 1$ for all natural n.This is easy to prove by math induction.
Really, $p_1 - 2q_1 = 9 - 8 = 1$ and since
$p_{n+1} - 2q_{n+1} = (2q_n + p_n)^2 - 8p_nq_n = (2q_n - p_n)^2 = (p_n - 2q_n)^2$
from supposition $2q_n - p_n$ follows $p_{n+1} - 2q_{n+1} = 1$.
Now using math. induction we will prove that q_n and $2q_n + 1$ are a perfect squares for any natural n.
Really, $q_1 = 4 = 2^2, 2q_1 + 1 = 9 = 3^2$.
Suppose now that $q_n = m^2$ and $2q_n + 1 = k^2$ for some natural m, k.
Then, since $q_{n+1} = 4q_np_n = 4q_n(2q_n + 1)$ we obtain $q_{n+1} = (2mk)^2$and
$2q_{n+1} + 1 = 8q_n(2q_n + 1) + 1 = (4q_n + 1)^2$.

Problem 7.14.

Since $a_n > 0$ for all n we will consider equivalent recurrence

$$a_{n+1}^2 = \frac{a_n^2}{4} + 1 + \frac{1}{a_n^2} \iff a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n},$$

which by substitution $a_n = \sqrt{\dfrac{p_n}{q_n}}$ can be rewritten in the form

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{4q_n} + 1 + \frac{q_n}{p_n} = \frac{(2q_n + p_n)^2}{4p_n q_n}.$$

Thus we can consider two recurrences

$p_{n+1} = (2q_n + p_n)^2$, $q_{n+1} = 4p_n q_n$, $n = 1, 2, ...,$ where $p_1 = 9, q_1 = 4,$ since $a_1 = \dfrac{3}{2}$.

Let $b_n := \dfrac{2}{\sqrt{a_n^2 - 2}}$, then $b_n^2 = \dfrac{4}{a_n^2 - 2} = \dfrac{4q_n}{p_n - 2q_n}$ is integer because $p_n - 2q_n = 1$ for all natural n.This is easy to prove by math induction.

Really, $p_1 - 2q_1 = 9 - 8 = 1$ and since

$p_{n+1} - 2q_{n+1} = (2q_n + p_n)^2 - 8p_n q_n = (2q_n - p_n)^2 = (p_n - 2q_n)^2$

from supposition $2q_n - p_n$ follows $p_{n+1} - 2q_{n+1} = 1$.

Now using math. induction we will prove that q_n and $2q_n + 1$ are a perfect squares for any natural n.

Really, $q_1 = 4 = 2^2, 2q_1 + 1 = 9 = 3^2$.

Suppose now that $q_n = m^2$ and $2q_n + 1 = k^2$ for some natural m, k.

Then, since $q_{n+1} = 4q_n p_n = 4q_n (2q_n + 1)$ we obtain $q_{n+1} = (2mk)^2$and

$2q_{n+1} + 1 = 8q_n (2q_n + 1) + 1 = (4q_n + 1)^2$.

Problem 7.15(All Israel Math. Olympiad in Hayfa)

Suppose opposite,i.e. that there is sequence $r_1 < r_2 < ... < r_k < ...$ of natural numbers such that $A_{r_k} = 0$. All this $r_k, k = 1, 2, 3, ...$ should be odd numbers, because, otherwise, all terms of sum A_{r_k} are positive and then $A_{r_k} > 0$.

Since $r_k, k = 1, 2, 3, ...$ odd numbers then from $A_{r_k} = 0$ follows that sum A_{r_k} must contain positive and negative numbers, i.e. in supposition that

$a_i = \begin{cases} b_i, i = 1, 2, ..., p \\ -b_i, i = p + 1, p + 2, ..., m \end{cases}$, we have $A_{r_k} = \sum\limits_{i=1}^{p} b_i^{r_k} - \sum\limits_{i=p+1}^{m} b_i^{r_k}$.

Thus, $A_{r_k} = 0 \iff b_1^{r_k} + b_2^{r_k} + ... + b_p^{r_k} = b_{p+1}^{r_k} + b_{p+2}^{r_k} + ... + b_m^{r_k}$.

There are $i \in \{1, 2, ..., p\}$ and $j \in \{p + 1, ..., m\}$ for which $b_i^{r_k} = b_j^{r_k}$ because otherwise, we get contradiction.

(Really, without loss of generality we can suppose that

$b_1 = \max\{b_1, b_2, ..., b_m\}$ and then, denoting $c_i := \dfrac{b_i}{b_1}, i = 2, 3, ..., m,$

we obtain that $1 + \sum\limits_{i=2}^{p} c_i^{r_k} = \sum\limits_{i=p+1}^{m} c_i^{r_k}$.

Since $0 < c_i < 1, i = 2, 3, ..., m$ then $\lim\limits_{k\to\infty} c_i^{r_k} = 0$ and, therefore,

$$1 = \lim_{k\to\infty} \left(1 + \sum_{i=2}^{p} c_i^{r_k}\right) = \lim_{k\to\infty} \sum_{i=p+1}^{m} c_i^{r_k} = 0.)$$

Using this property we obtain the same situation for $m - 2$ numbers and after $m - 4$ and so on till $m - 2k > 0$,i.e. till we get one non-zero number which equal to zero.

Remark.
The statement "There are $i \in \{1,2,...,p\}$ and $j \in \{p+1,...,m\}$ for which $b_i^{r_k} = b_j^{r_k}$" can be proved shortly by the such way:
Since $b_1^{r_k} + b_2^{r_k} + ... + b_p^{r_k} = b_{p+1}^{r_k} + b_{p+2}^{r_k} + ... + b_m^{r_k}$ then $\max\{b_1, b_2, ..., b_p\} = \lim_{k\to\infty} \sqrt[r_k]{b_1^{r_k} + b_2^{r_k} + ... + b_p^{r_k}} = \lim_{k\to\infty} \sqrt[r_k]{b_{p+1}^{r_k} + b_{p+2}^{r_k} + ... + b_m^{r_k}} = \max\{b_{p+1}, b_{p+2}, ..., b_m\}$.

Problem 7.16*(#7,9-th grade,18-th All Soviet Union Math Olympiad,1984) (Proposed by Agahanov N.H.)
Let us calculate several first terms of the given sequence:
$x_1 = 1, x_2 = -1, x_3 = \dfrac{1}{2}, x_4 = \dfrac{3}{4}, x_5 = \dfrac{5}{16}, x_6 = -\dfrac{71}{256}$
Lemma.
Let sequence (p_n) determined by recurrence $p_{n+1} = p_n^2 + \dfrac{p_n}{2}, n \in \mathbb{N}$ with initial
condition $p_1 = \dfrac{3}{8}$. Then for any $n \in \mathbb{N}$ holds following inequalities:
i. $p_n < \dfrac{1}{2}, n \in \mathbb{N}$;
ii. $p_n > p_{n+1}, n \in \mathbb{N}$;
iii. $\max\{|x_{2n+3}|, |x_{2n+4}|\} \le p_n$.
Proof.(Math. Induction by n)
1. Base of induction.
Let $n = 1$ then $p_1 = \dfrac{3}{8} < \dfrac{1}{2},\ p_2 = \dfrac{3}{8}\left(\dfrac{3}{8} + \dfrac{1}{2}\right) = \dfrac{21}{64} < \dfrac{3}{8} = p_1$
and $\max\{|x_5|, |x_6|\} = \max\left\{\dfrac{5}{16}, \dfrac{71}{256}\right\} = \dfrac{5}{16} < \dfrac{3}{8} = p_1$.
2. Step of induction.
i. $p_{n+1} = p_n^2 + \dfrac{p_n}{2} < \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2}$;
ii. $p_{n+1} = p_n^2 + \dfrac{p_n}{2} < p_n \iff p_n < \dfrac{1}{2}$.
iii. $|x_{2n+5}| \le |x_{2n+4}|^2 + \dfrac{1}{2}|x_{2n+3}| \le p_n^2 + \dfrac{p_n}{2} = p_{n+1}$ and
$|x_{2n+6}| \le |x_{2n+5}|^2 + \dfrac{1}{2}|x_{2n+4}| \le p_{n+1}^2 + \dfrac{p_n}{2} \le p_n^2 + \dfrac{p_n}{2} = p_{n+1}$.
Corollary1.
$\lim_{n\to\infty} p_n = 0$.
Corollary 2
$\lim_{n\to\infty} x_n = 0$.

Problem 7.17

$$a_{n+1} \le a_n(1-a_n) \iff \frac{1}{a_{n+1}} \ge \frac{1}{a_n(1-a_n)} = \frac{1}{a_n} + \frac{1}{1-a_n} \iff$$

$$\frac{1}{a_{n+1}}-\frac{1}{a_n}=\frac{(1-a_n)+a_n}{1-a_n}=1+\frac{a_n}{1-a_n}=1+\frac{a_n^2}{a_n(1-a_n)}=1+\frac{a_n^2}{a_{n+1}}>1.$$

Thus, $\sum\limits_{k=1}^{n}\left(\dfrac{1}{a_{k+1}}-\dfrac{1}{a_k}\right)>n \iff \dfrac{1}{a_{n+1}}-\dfrac{1}{a_1}>n \implies$
$\dfrac{1}{a_n}>n-1+\dfrac{1}{a_1}=\dfrac{a_1(n-1)+1}{a_1} \iff a_n<\dfrac{a_1}{a_1(n-1)+1} \implies$
$na_n<\dfrac{na_1}{a_1(n-1)+1}<2$ since $\dfrac{na_1}{a_1(n-1)+1}<2 \iff$
$na_1<2a_1(n-1)+2 \iff 2a_1-2<2a_1n.$

Problem 7.18(BAMO-2000)

Since $a_n(1-a_n)\le\dfrac{1}{4}$ and $a_n^2\le a_n-a_{n+1} \iff a_{n+1}\le a_n(1-a_n)$ then
$a_n\in\left(0,\dfrac{1}{4}\right)$ and $\dfrac{1}{a_{n+1}}\ge\dfrac{1}{a_n}+\dfrac{1}{1-a_n}>\dfrac{1}{a_n}+1$ for any $n\ge 2$
because $\dfrac{1}{1-a_n}>1.$

Thus, for any $n\ge 2$ we have $\dfrac{1}{a_{n+1}}>\dfrac{1}{a_n}+1 \iff$
$\dfrac{1}{a_{n+1}}-(n+1)>\dfrac{1}{a_n}-n \implies \dfrac{1}{a_n}-n>\dfrac{1}{a_2}-2\ge 4-2=2 \implies$
$\dfrac{1}{a_n}>n+2 \iff a_n<\dfrac{1}{n+2}<\dfrac{1}{n}.$

Problem 7.19 (SSMJ 5281)

First note that $a_n>0$ for all $n\in\mathbb{N}$ ($a_1=a>0$ and from supposition
$a_n>0$ follows $a_{n+1}=\dfrac{a_n}{1+a_n^p}>0$. Also note that sequence $\{a_n\}_{n\ge1}$
is decreasing. Indeed $a_n-a_{n+1}=a_n-\dfrac{a_n}{1+a_n^p}=\dfrac{a_n^{p+1}}{1+a_n^p}>0.$
Therefore, sequence $\{a_n\}_{n\ge1}$ convergent to some nonnegative limit x.
Then $x=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\dfrac{a_n}{1+a_n^p}=\dfrac{x}{1+x^p} \implies x=0.$
Thus, $\lim\limits_{n\to\infty}a_n=0.$
Since recurrence $a_{n+1}=\dfrac{a_n}{1+a_n^p}$ can be rewritten in the form
$a_{n+1}^p=\dfrac{a_n^p}{(1+a_n^\alpha)^p}$,then denoting a_n^p via b_n we obtain recurrence

(1) $\qquad b_{n+1}=\dfrac{b_n}{(1+b_n)^p}$,with initial condition $b_1=a^p$.

Since $\dfrac{1}{b_{n+1}}-\dfrac{1}{b_n}=\dfrac{(1+b_n)^p-1}{b_n}$ and $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}a_n^p=0$ then $\lim\limits_{n\to\infty}\left(\dfrac{1}{b_{n+1}}-\dfrac{1}{b_n}\right)=$
$\lim\limits_{n\to\infty}\dfrac{(1+b_n)^p-1}{b_n}=p$. Hereof, by Arithmetic Mean Limit Theorem
(if $\lim\limits_{n\to\infty}x_n=a$ then $\lim\limits_{n\to\infty}\dfrac{x_1+x_2+...+x_n}{n}=a$) we obtain

$\lim\limits_{n\to\infty}\dfrac{1}{nb_n}=\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{b_n}-\dfrac{1}{b_1}}{n-1}\cdot\dfrac{n-1}{n}=\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=2}^{n}\left(\dfrac{1}{b_k}-\dfrac{1}{b_{k-1}}\right)}{n-1}=$
$\lim\limits_{n\to\infty}\left(\dfrac{1}{b_n}-\dfrac{1}{b_{n-1}}\right)=p.$

Thus, $\lim\limits_{n\to\infty}n^{\frac{1}{p}}a_n=\lim\limits_{n\to\infty}(na_n^p)^{\frac{1}{p}}=\lim\limits_{n\to\infty}(nb_n)^{\frac{1}{p}}=\left(\dfrac{1}{p}\right)^{\frac{1}{p}}$ and,
therefore, $\lim\limits_{n\to\infty}\dfrac{a_n}{\left(\dfrac{1}{np}\right)^{\frac{1}{p}}}a_n=1.$

Hence, $\sum\limits_{n=1}^{\infty}a_n$ is convergent iff $\sum\limits_{n=1}^{\infty}\dfrac{1}{(np)^{\frac{1}{p}}}$ is convergent,

that is iff $\dfrac{1}{p}>1\iff p<1.$

Problem 7.20
We can see that sequence $2\cosh(2^n\alpha)$ where $\alpha:=\cosh^{-1}(2.5)$
satisfy to the recurrence $a_{n+1}=a_n^2-2$,since
$2\left(2\cosh^2(2^n\alpha)-1\right)=2\cosh\left(2^{n+1}\alpha\right)$ and $2\cosh\left(2^0\alpha\right)=5$.
Thus, $a_n=2\cosh(2^n\alpha)$.
a) Since $2\sinh t\cdot\cosh t=\sinh 2t$ then
$\dfrac{a_{n+1}}{a_1a_2...a_n}=\dfrac{2\cosh\left(2^{n+1}\alpha\right)}{2^n\cosh(2\alpha)\cosh(2\alpha)...\cosh(2^n\alpha)}\cdot\dfrac{\sinh(2\alpha)}{\sinh(2\alpha)}=$
$\dfrac{2\cosh\left(2^{n+1}\alpha\right)\cdot\sinh(2\alpha)}{\sinh(2^{n+1}\alpha)}=2\sinh(2\alpha)\cdot\coth\left(2^{n+1}\alpha\right).$
Note that $\lim\limits_{n\to\infty}\dfrac{\cosh\left(2^{n+1}\alpha\right)}{\sinh(2^{n+1}\alpha)}=\lim\limits_{n\to\infty}\dfrac{e^{2^{n+1}\alpha}+e^{-2^{n+1}\alpha}}{e^{2^{n+1}\alpha}-e^{-2^{n+1}\alpha}}=1$ because $\alpha>0$.
Hence, $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_1a_2...a_n}=2\sinh(2\alpha)=2\sqrt{\cosh^2(2\alpha)-1}=$
$2\sqrt{\dfrac{25}{4}-1}=\sqrt{21}.$
b) Note that initial recurrence can be rewritten in the form:
$$1-\frac{a_n^2}{2}=-\frac{a_{n+1}}{2},\ n\in\mathbb{N}.$$
Using that we obtain:
$\dfrac{1}{a_1}+\dfrac{1}{a_1a_2}+...+\dfrac{1}{a_1a_2...a_n}-\dfrac{a_1}{2}=\dfrac{1}{a_1}\left(1-\dfrac{a_1^2}{2}+\dfrac{1}{a_2}+\dfrac{1}{a_2a_3}+...+\dfrac{1}{a_1a_2...a_n}\right)=$
$\dfrac{1}{a_1}\left(\dfrac{1}{a_2}+\dfrac{1}{a_2a_3}+...+\dfrac{1}{a_2...a_n}-\dfrac{a_2}{2}\right)=\dfrac{1}{a_1a_2}\left(1-\dfrac{a_2^2}{2}+\dfrac{1}{a_3}+...+\dfrac{1}{a_3...a_n}\right)=$
$\dfrac{1}{a_1a_2}\left(\dfrac{1}{a_3}+...+\dfrac{1}{a_3...a_n}-\dfrac{a_3}{2}\right)=...=\dfrac{1}{a_1a_2...a_n}\left(1-\dfrac{a_n^2}{2}\right)=-\dfrac{a_{n+1}}{a_1a_2...a_n}.$
Since that $\lim\limits_{n\to\infty}\left(\dfrac{1}{a_1}+\dfrac{1}{a_1a_2}+...+\dfrac{1}{a_1a_2...a_n}\right)=\lim\limits_{n\to\infty}\left(\dfrac{a_1}{2}-\dfrac{a_{n+1}}{a_1a_2...a_n}\right)=$

$\cosh 2\alpha - 2\sinh 2\alpha = \dfrac{5}{2} - \sqrt{21}.$

Remark.

Another solutions :

We will consider this problem in general case, when

$$a_0 = a, a_{n+1} = a_n^2 - 2, n \in \mathbb{N} \cup \{0\}$$

and a is any real number greater then 2.

1.Then $a_n = 2\cosh(2^n\alpha)$ where $\alpha = \cosh^{-1}\left(\dfrac{a}{2}\right) = \ln\left(\dfrac{a+\sqrt{a^2-4}}{2}\right)$.

Denote $S(a_1, a_2, ..., a_n) := \dfrac{1}{a_1} + \dfrac{1}{a_1a_2} + ... + \dfrac{1}{a_1a_2...a_n}$.

Since $\dfrac{a_{k+1}}{a_1a_2...a_k} - \dfrac{a_k}{a_1a_2...a_{k-1}} = \dfrac{1}{a_1a_2...a_k}\left(a_{k+1} - a_k^2\right) = -\dfrac{2}{a_1a_2...a_k}$

and $\prod_{k=1}^{0} a_k = 1$ then $\dfrac{1}{a_1a_2...a_k} = \dfrac{a_k}{2a_1a_2...a_{k-1}} - \dfrac{a_{k+1}}{2a_1a_2...a_k}$ and

$S(a_1, a_2, ..., a_n) = \dfrac{a_1}{2} - \dfrac{a_{n+1}}{2a_1a_2...a_n}$.

Since $\dfrac{a_{n+1}}{a_1a_2...a_n} = 2\sinh(2\alpha)\cdot\coth\left(2^{n+1}\alpha\right)$ then

$S(a_1, a_2, ..., a_n) = \dfrac{2\cosh(2\alpha) - 2\sinh(2\alpha)\cdot\coth\left(2^{n+1}\alpha\right)}{2} =$

$\cosh(2\alpha) - \sinh(2\alpha)\cdot\coth\left(2^{n+1}\alpha\right)$.

and, using $\lim\limits_{n\to\infty}\coth\left(2^{n+1}\alpha\right) = 1,$ we finally obtain

$$\lim_{n\to\infty}\left(\frac{1}{a_1} + \frac{1}{a_1a_2} + ... + \frac{1}{a_1a_2...a_n}\right) = \cosh(2\alpha) - \sinh(2\alpha) =$$

$$\cosh^2\alpha + \sinh^2\alpha - 2\cosh\alpha\sinh\alpha = (\cosh\alpha - \sinh\alpha)^2 .$$

2. Both solutions above short but bad motivated. The following solution I like more, because it is motivated solution.

First note that infinite sum $\dfrac{1}{a_k} + \dfrac{1}{a_ka_{k+1}} + ... + \dfrac{1}{a_ka_{k+1}...a_n} + ...$ converge, because increasing sequence $(S(a_k, a_{k+1}, ..., a_n))_{n\ge k}$ have upper bound.

Indeed, since a_n increasing, then

$$\frac{1}{a_k} + \frac{1}{a_ka_{k+1}} + ... + \frac{1}{a_ka_{k+1}...a_n} < \frac{1}{a_k} + \frac{1}{a_k^2} + ... + \frac{1}{a_k^{n-k+1}} < \frac{\frac{1}{a_k}}{1 - \frac{1}{a_k}} = \frac{1}{a_k - 1} < \frac{1}{a-1}.$$

Since $a_k = 2\cosh\left(2^k\alpha\right)$ and infinite sum

$$\frac{1}{a_k} + \frac{1}{a_ka_{k+1}} + ... + \frac{1}{a_ka_{k+1}...a_n} + ...$$

depend only from a_k then we can denote this sum via $S\left(2^k\alpha\right)$.

Thus, $\dfrac{1}{a_1} + \dfrac{1}{a_1a_2} + ... + \dfrac{1}{a_1a_2...a_n} + ... = S(\varphi)$ and

$\dfrac{1}{a_2} + \dfrac{1}{a_2a_3} + ... + \dfrac{1}{a_2a_3...a_n} + ... = S(2\varphi)$,where $\varphi := 2\alpha$.

Since $S(\varphi)=\dfrac{1}{a_1}(1+S(2\varphi)) \iff S(2\varphi)=a_1S(\varphi)-1 \iff$
$S(2\varphi)=2\cosh\varphi\cdot S(\varphi)-1$ then our problem now is to find solution of this functional equation.
First note that since $\cosh 2\varphi=2\cosh^2\varphi-1$ then $h(\varphi):=S(\varphi)-\cosh\varphi$ satisfy to homogeneous linear functional equation $h(2\varphi)=2\cosh\varphi\cdot h(\varphi)$.
Using representation $h(\varphi)$ in the form $h(\varphi)=C(\varphi)\cdot\sinh\varphi$ and identity $\sinh(2\varphi)=2\cosh\varphi\cdot\sinh\varphi$ we obtain
$C(2\varphi)\cdot\sinh 2\varphi=2\cosh\varphi\cdot C(\varphi)\cdot\sinh\varphi \iff$
$C(2\varphi)=C(\varphi) \implies C(\varphi)=C\left(\dfrac{\varphi}{2^n}\right), n\in\mathbb{N}.$
In the supposition that $C(\varphi)$ is continuous function (series $S(\varphi)$ converges uniformly) we immediately obtain that $C(\varphi)=C=const$.
So, $h(\varphi)=C\cdot\sinh\varphi$ and $S(\varphi)=\cosh\varphi+C\cdot\sinh\varphi$.
Since $S(2^n\varphi)=\dfrac{1}{a_{n+1}}+\dfrac{1}{a_{n+1}a_{n+2}}+.....<\dfrac{1}{a_{n+1}-1}$ then
$\lim\limits_{n\to\infty}S(2^n\varphi)=0 \iff C=-\lim\limits_{n\to\infty}\coth 2^n\varphi=-1.$
Thus, $S(\varphi)=\cosh\varphi-\sinh\varphi=\cosh 2\alpha-\sinh 2\alpha=(\cosh\alpha-\sinh\alpha)^2$ and since
$\cosh\alpha=\dfrac{a}{2},\ \sinh\alpha=\sqrt{\dfrac{a^2}{4}-1}=\dfrac{\sqrt{a^2-4}}{2}$ then
$S(\varphi)=\dfrac{\left(a-\sqrt{a^2-4}\right)^2}{4}$ if $a_0=a$.
If $a_1=a$ then $\cosh 2\alpha=\dfrac{a}{2},\ \sinh 2\alpha=\dfrac{\sqrt{a^2-4}}{2}$ and
$$S(\varphi)=\cosh\varphi-\sinh\varphi=\frac{a-\sqrt{a^2-4}}{2}.$$

Problem 7.21*
a) Note that (a_n) decreasing sequence $\left(a_{n+1}=\dfrac{a_n}{1+\sqrt{a_n}}<a_n, n\in\mathbb{N}\right)$.
Then in particularly $a_n\le a_1=a$, $n\in\mathbb{N}$ and since
$a_{n+1}=\dfrac{a_n}{1+\sqrt{a_n}} \iff \dfrac{1}{a_{n+1}}=\dfrac{1}{a_n}+\dfrac{1}{\sqrt{a_n}} \iff \dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}=\dfrac{1}{\sqrt{a_n}}$
we obtain $\dfrac{1}{a_{n+1}}-\dfrac{1}{a_1}=\sum\limits_{k=1}^{n}\left(\dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}\right)=\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{a_k}}\ge$
$\dfrac{n}{\sqrt{a_1}} \iff \dfrac{1}{a_{n+1}}\ge\dfrac{1}{a^2}+\dfrac{n}{a} \implies \dfrac{1}{a_{n+1}}\ge\dfrac{n}{a}\ge\dfrac{n+1}{2a}.$
Thus, $a_n\le\dfrac{2a}{n}$for any $n\ \ge 2$.
It is not enough for the proof that S_n is bounded, but, using inequality $a_n\le\dfrac{2a}{n} \iff \dfrac{1}{a_n}\ge\dfrac{n}{2a}$ and identity
$\dfrac{1}{a_{n+1}}=\dfrac{1}{a_1}+\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{a_k}},$
we can obtain better upper bound for a_n.
Really, since $a_n\le\dfrac{2a}{n}$ for any natural $n\ge 2$, we have

$$\frac{1}{a_{n+1}} = \frac{1}{a^2} + \frac{1}{a} + \sum_{k=2}^{n} \frac{1}{\sqrt{a_k}} \geq \frac{1}{a^2} + \frac{1}{a} + \sum_{k=2}^{n} \frac{\sqrt{k}}{\sqrt{2a}} = \frac{1}{a^2} + \frac{1}{a} + \frac{1}{\sqrt{2a}} \sum_{k=1}^{n-1} \sqrt{k+1} \, .$$

From the other hand, for any natural n holds inequality

(1) $\quad \frac{3}{2}\sqrt{n+1} \geq (n+1)\sqrt{n+1} - n\sqrt{n}$, $n \in \mathbb{N}$.

(Since $(n+1)\sqrt{n+1} - n\sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\left(2n+1+\sqrt{n(n+1)}\right) =$
$\dfrac{2n+1+\sqrt{n(n+1)}}{\sqrt{n+1}+\sqrt{n}}$ then $\frac{3}{2}\sqrt{n+1} \geq (n+1)\sqrt{n+1} - n\sqrt{n} \iff$
$3\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right) \geq 2\left(2n+1+\sqrt{n(n+1)}\right) \iff$
$3n+3+3\sqrt{n(n+1)} \geq 4n+2+2\sqrt{n(n+1)} \iff \sqrt{n(n+1)} \geq n-1$).

Using inequality **(1)** we obtain $\dfrac{1}{a_{n+1}} \geq \dfrac{1}{a^2}+\dfrac{1}{a}+\dfrac{1}{\sqrt{2a}}\sum_{k=1}^{n-1}\sqrt{k+1} \geq \dfrac{1}{a^2}+\dfrac{1}{a}+$
$\dfrac{2}{3\sqrt{2a}}\sum_{k=1}^{n-1}\left((k+1)\sqrt{k+1}-k\sqrt{k}\right) = \dfrac{1}{a^2}+\dfrac{1}{a}+\dfrac{2}{3\sqrt{2a}}\left(n\sqrt{n}-1\right) =$
$\dfrac{1}{a}+\dfrac{1}{\sqrt{a}}-\dfrac{2}{3\sqrt{2a}}+\dfrac{2}{3\sqrt{2a}}\left(n\sqrt{n}-1\right)$ and since $\dfrac{1}{a}+\dfrac{1}{\sqrt{a}}-\dfrac{2}{3\sqrt{2a}} > 0$
then $\dfrac{1}{a_{n+1}} > cn\sqrt{n}$,where $c = \dfrac{2}{3\sqrt{2a}}$.

Thus, for any $n \geq 3$ holds $a_n \leq \dfrac{1}{c(n-1)\sqrt{n-1}}$ and, therefore,

$S_n \leq a_1 + a_2 + \sum_{k=3}^{n} \dfrac{1}{c(k-1)\sqrt{k-1}} = a_1 + a_2 + \dfrac{1}{c}\sum_{k=2}^{n-1}\dfrac{1}{k\sqrt{k}}$.

Since $\dfrac{1}{k\sqrt{k}} < \dfrac{2}{k\sqrt{k-1}+(k-1)\sqrt{k}} = \dfrac{2}{\sqrt{k-1}} - \dfrac{2}{\sqrt{k}}$
($k\sqrt{k-1}+(k-1)\sqrt{k} \leq 2k\sqrt{k} \iff k\sqrt{k-1} \leq k\sqrt{k}+\sqrt{k} = \sqrt{k}(k+1) \iff$

$k^3 - k^2 \leq k^3 + 2k^2 + k$) for $k \geq 2$ then $\sum_{k=2}^{n-1}\dfrac{1}{k\sqrt{k}} = \dfrac{2}{\sqrt{1}} - \dfrac{2}{\sqrt{n-1}} < 2$.

and $a_1 + a_2 + \dfrac{2}{c}$ is upper bound for S_n .

b) Since $\sqrt{a_{n+1}} = \dfrac{\sqrt{a_n}}{\sqrt{1+\sqrt{a_n}}}$ then by setting $b_n := \sqrt{a_n}$ we obtain

for sequence (b_n) recurrence $b_{n+1} = \dfrac{b_n}{\sqrt{1+b_n}}$ with initial condition
$b_1 = 3$ and we will attempt to find "good" bounds for b_n.
If we get success, then, square of this bounds becames "good"
bounds for a_n.

Since $\dfrac{1}{b_{n+1}} = \dfrac{\sqrt{1+b_n}}{b_n} \iff \dfrac{1}{b_{n+1}} = \dfrac{1}{b_n} + \dfrac{\sqrt{1+b_n}-1}{b_n} \iff$
$\dfrac{1}{b_{n+1}} - \dfrac{1}{b_n} = \dfrac{1}{\sqrt{1+b_n}+1}$

then $\dfrac{1}{b_{n+1}} - \dfrac{1}{b_1} = \sum_{k=1}^{n}\left(\dfrac{1}{b_{k+1}} - \dfrac{1}{b_k}\right) = \sum_{k=1}^{n}\dfrac{1}{\sqrt{1+b_k}+1} \implies$

(2) $\qquad \dfrac{1}{b_{n+1}} = \dfrac{1}{3} + \sum_{k=1}^{n}\dfrac{1}{\sqrt{1+b_k}+1}.$

Since b_n decreasing then $b_n \le 3, n \in \mathbb{N}$ and from **(2)** immediately follows inequality

$$\frac{1}{3} + \sum_{k=1}^{n}\frac{1}{\sqrt{1+3}+1} \le \frac{1}{b_{n+1}} \iff \frac{1}{3} + \frac{n}{3} \le \frac{1}{b_{n+1}} \implies \frac{n+1}{3} \le \frac{1}{b_{n+1}} \implies \frac{n}{3} \le \frac{1}{b_n} \iff b_n \le \frac{3}{n}.$$

Using inequality $\sqrt{1+x} < 1 + \dfrac{x}{2}$ for $x > 0$ and inequality $b_n \le \dfrac{3}{n}, n \in \mathbb{N}$ we obtain:

$$\frac{1}{b_{n+1}} - \frac{1}{b_n} = \frac{1}{\sqrt{1+b_n}+1} > \frac{1}{1+\dfrac{b_n}{2}+1} = \frac{2}{b_n+4} > \frac{2}{\dfrac{3}{n}+4} = \frac{2n}{4n+3} =$$

$$\frac{1}{2}\cdot\frac{4n-3+3}{4n+3} = \frac{1}{2}\left(1 - \frac{3}{4n+3}\right) > \frac{1}{2}\left(1-\frac{1}{n}\right)$$ and, therefore,

$$\frac{1}{b_{n+1}} - \frac{1}{b_1} > \sum_{k=1}^{n}\frac{1}{2}\left(1-\frac{1}{k}\right) = \frac{n}{2} - \frac{1}{2}h_n,$$ where $h_n = 1 + \dfrac{1}{2} + ... + \dfrac{1}{n}$.

From the other hand, since $b_n > 0$ then $\dfrac{1}{b_{n+1}} - \dfrac{1}{b_1} = \sum_{k=1}^{n}\left(\dfrac{1}{b_{n+1}} - \dfrac{1}{b_n}\right) =$

$$\sum_{k=1}^{n}\frac{1}{\sqrt{1+b_k}+1} < \sum_{k=1}^{n}\frac{1}{\sqrt{1+0}+1} = \frac{n}{2}.$$

So, $\dfrac{n}{2} - \dfrac{1}{2}h_n < \dfrac{1}{b_{n+1}} - \dfrac{1}{b_1} < \dfrac{n}{2} \iff \dfrac{n}{2} - \dfrac{1}{2}h_n + \dfrac{1}{3} < \dfrac{1}{b_{n+1}} < \dfrac{1}{3} + \dfrac{n}{2}.$

Since $\dfrac{1}{3} + \dfrac{n}{2} < \dfrac{n+1}{2}$ and $\dfrac{n}{2} - \dfrac{1}{2}h_n + \dfrac{1}{3} < \dfrac{n}{2} - \dfrac{h_n}{2}$ then we obtain more convenient inequality

$$\frac{n-h_n}{2} < \frac{1}{b_{n+1}} < \frac{n+1}{2} \iff \frac{2}{n+1} < b_{n+1} < \frac{2}{n-h_n} \implies \frac{2}{n+1} < b_{n+1} < \frac{2}{n-h_{n+1}} \implies$$

(3) $\qquad \dfrac{2}{n} < b_n < \dfrac{2}{n-h_n-1}, n > 1.$

Since $\dfrac{h_n}{n} < \left(\dfrac{1+\dfrac{1}{2^2}+...+\dfrac{1}{n^2}}{n}\right)^{\frac{1}{2}}$ and

$$1 + \frac{1}{2^2} + ... + \frac{1}{n^2} < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} ... + \frac{1}{(n-1)\,n} = 1 + 1 - \frac{1}{n} < 2$$

then $\dfrac{h_n}{n} < \sqrt{\dfrac{2}{n}}$.

From this inequality follows that $h_n < \sqrt{2n}$ and $\lim\limits_{n\to\infty}\dfrac{h_n}{n} = 0.$

Since, $2 < nb_n < \dfrac{2}{1-\dfrac{1+h_n}{n}}$ and $\lim\limits_{n\to\infty}\dfrac{h_n}{n}=0$ then we obtain $\lim\limits_{n\to\infty} nb_n = 2.$

So, inequality **(3)** gives us good bounds for b_n: lover bound $l(n)=\dfrac{2}{n}$ and upper bound $u(n)=\dfrac{2}{n-h_n-1}$.

Since $h(n)<\sqrt{2n}$ then we can use more convenient lover bound for b_n, namely we can take $u(n)=\dfrac{2}{n-\sqrt{2n}-1}$.

Thus, for a_n we obtain inequality $\dfrac{4}{n^2}<a_n<\dfrac{4}{(n-h_n-1)^2}$ or, inequality

$$\frac{4}{n^2}<a_n<\frac{4}{\left(n-\sqrt{2n}-1\right)^2}<\frac{4}{n^2-2n\sqrt{2n}}<\frac{4}{n^2-3n\sqrt{n}}$$

which determine good bounds for a_n and asymptotic representation $a_n\sim\dfrac{4}{n^2}$.

c) Since recurrence $a_{n+1}=\dfrac{a_n}{1+\sqrt{a_n}}$ can be rewritten in the form

$\sqrt{a_{n+1}}=\dfrac{\sqrt{a_n}}{\sqrt{1+\sqrt{a_n}}}$,

then denoting $\sqrt{a_n}$ via b_n we obtain recurrence

(1) $\qquad b_{n+1}=\dfrac{b_n}{\sqrt{1+b_n}}$,with initial condition $b_1=\sqrt{a}$, $\;b^2-1$

with the same question about good bounds for sequence (b_n).

This is the way to solve the original problem, because sequence (b_n) more convenient object to give answer on question of problem.

For convenience we set $a:=\left(b^2-1\right)^2$, where $b>1$.Then $b_1:=b^2-1$.

Note that from recurrence **(1)** obviously follows, that b_n decreasing in $\mathbb{N}$ ($a_{n+1}=\dfrac{a_n}{1+\sqrt{a_n}}<a_n, n\in\mathbb{N}$). In particularly this yields $b_n\le b_1=b^2-1$.

Let us rewrite recurrence (*) in the form important for further:

$\dfrac{1}{b_{n+1}}=\dfrac{\sqrt{1+b_n}}{b_n}\iff\dfrac{1}{b_{n+1}}=\dfrac{1}{b_n}+\dfrac{\sqrt{1+b_n}-1}{b_n}\iff$

(2) $\qquad \dfrac{1}{b_{n+1}}-\dfrac{1}{b_n}=\dfrac{1}{\sqrt{1+b_n}+1}$.

Hereof we obtain correlation

(3) $\qquad \sum\limits_{k=1}^{n}\left(\dfrac{1}{b_{k+1}}-\dfrac{1}{b_k}\right)=\dfrac{1}{b_{n+1}}-\dfrac{1}{b_1}=\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{1+b_n}+1}$.

From **(3)** and $b_n\le b^2-1$ follows

$\dfrac{1}{b_{n+1}}-\dfrac{1}{b_1}=\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{1+b_k}+1}\ge\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{1+(b^2-1)}+1}=\dfrac{n}{b+1}\implies$

$\dfrac{1}{b_{n+1}}\ge\dfrac{1}{b_1}+\dfrac{n}{b+1}=\dfrac{n(b-1)+1}{b^2-1}$

and since $\dfrac{n(b-1)+1}{b^2-1}>\dfrac{n+1}{b(b+1)}$ for any natural n ,we obtain

$\dfrac{1}{b_{n+1}} > \dfrac{n+1}{b(b+1)} \iff b_{n+1} < \dfrac{b(b+1)}{n+1} \implies b_n < \dfrac{b(b+1)}{n}.$

Since for any $x > 0$ holds inequality $\sqrt{1+x} < 1 + \dfrac{x}{2}$ and

$b_n \in \left(0, \dfrac{b(b+1)}{n}\right)$ for any $n \in \mathbb{N}$ then

$\dfrac{1}{1+\dfrac{b_n}{2}+1} < \dfrac{1}{\sqrt{1+b_n}+1} < \dfrac{1}{\sqrt{1+0}+1} \iff$

(4) $\dfrac{2}{b_n+4} < \dfrac{1}{\sqrt{1+b_n}+1} < \dfrac{1}{2}$ and

$\dfrac{2}{b_n+4} > \dfrac{2}{\dfrac{b(b+1)}{n}+4} = \dfrac{2n}{4n+b(b+1)} =$

$\dfrac{1}{2}\left(1 - \dfrac{b(b+1)}{4n+b(b+1)}\right) > \dfrac{1}{2}\left(1 - \dfrac{b(b+1)}{4n}\right).$

Thus, for any $n \in \mathbb{N}$ holds inequality

(5) $\dfrac{1}{2}\left(1 - \dfrac{b(b+1)}{4n}\right) < \dfrac{1}{\sqrt{1+b_n}+1} < \dfrac{1}{2}.$

Using **(5)** and **(3)** we obtain

$\dfrac{1}{2}\left(n - \dfrac{b(b+1)}{4} \cdot h_n\right) < \dfrac{1}{b_{n+1}} - \dfrac{1}{b_1} < \dfrac{n}{2} \iff$

(6) $\dfrac{1}{2}\left(n - \dfrac{b(b+1)}{4} \cdot h_n\right) + \dfrac{1}{b_1} < \dfrac{1}{b_{n+1}} < \dfrac{n}{2} + \dfrac{1}{b_1}$,

where $h_n = 1 + \dfrac{1}{2} + ... + \dfrac{1}{n}.$

Since $\dfrac{h_n}{n} < \left(\dfrac{1 + \dfrac{1}{2^2} + ... + \dfrac{1}{n^2}}{n}\right)^{\frac{1}{2}}$ and

$1 + \dfrac{1}{2^2} + ... + \dfrac{1}{n^2} < 1 + \dfrac{1}{1 \cdot 2} + ... + \dfrac{1}{(n-1)n} =$

$1 + 1 - \dfrac{1}{n} < 2$ then $\dfrac{h_n}{n} < \sqrt{\dfrac{2}{n}}$.

From this inequality follows that $h_n < \sqrt{2n}$ and $\lim\limits_{n\to\infty} \dfrac{h_n}{n} = 0.$

Let $c := \max\left\{0, \dfrac{1}{b_1} - \dfrac{1}{2}\right\}$ then $\dfrac{n}{2} + \dfrac{1}{b_1} \le \dfrac{n+1+c}{2}$ and

from **(6)** follows inequalities

$$\frac{1}{2}\left(n - \frac{b(b+1)}{4} \cdot h_n - 1\right) < \frac{1}{2}\left(n - 1 - \frac{b(b+1)}{4} \cdot h_{n-1}\right) < \frac{1}{b_n} < \frac{n+c}{2}$$

which implies

$\dfrac{1}{2}\left(1 - \dfrac{b(b+1)}{4} \cdot \dfrac{h_n}{n} - \dfrac{1}{n}\right) < \dfrac{1}{nb_n} < \dfrac{1}{2}\left(1 + \dfrac{c}{n}\right).$

Since $\lim_{n\to\infty} \frac{1}{2}\left(1-\frac{b(b+1)}{4}\cdot\frac{h_n}{n}-\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{c}{n}\right)=\frac{1}{2}$
then $\lim_{n\to\infty}\frac{1}{nb_n}=\frac{1}{2}$ as well.
Thus, $\lim_{n\to\infty} nb_n=2$ and we finally obtain that $\lim_{n\to\infty} n^2a_n=4$.

Problem 7.22 (One asymptotic behavior)(S183)
Let $x_n=\frac{p_n}{2}, n\in\mathbb{N}\cup\{0\}$ then $p_n=p_{n-1}-\frac{p_{n-1}^2}{2} \iff$
$x_n=x_{n-1}-x_{n-1}^2, n\in\mathbb{N}$ and $x_0=\frac{p_0}{2}=\frac{1}{p}$, where $p>1$.
Since $0<x_n, n\in\mathbb{N}\cup\{0\}$ ($x_0\in(0,1)$ and
$x_{n-1}\in(0,1) \implies x_n=x_{n-1}(1-x_{n-1})\in(0,1)$)
then $\frac{1}{x_n}-\frac{1}{x_0}=\sum_{k=1}^{n}\left(\frac{1}{x_k}-\frac{1}{x_{k-1}}\right)=$
$\sum_{k=1}^{n}\frac{1}{1-x_{k-1}}>\sum_{k=1}^{n}\frac{1}{1-0}=n \iff \frac{1}{x_n}>.p+n \iff x_n<\frac{1}{n+p}, n\in\mathbb{N}$.
Moreover, since $x_0=\frac{1}{p}$ then $x_n\leq\frac{1}{n+p}$, $n\geq 0$ and
for $n\geq 1$ we obtain

$$\frac{1}{x_n}-\frac{1}{x_0}=\sum_{k=1}^{n}\left(\frac{1}{x_k}-\frac{1}{x_{k-1}}\right)=\sum_{k=1}^{n}\frac{1}{1-x_{k-1}}\leq$$
$$\sum_{k=1}^{n}\frac{1}{1-\frac{1}{k-1+p}}=\sum_{k=1}^{n}\left(1+\frac{1}{k-2+p}\right)=$$
$$n+\sum_{k=1}^{n}\frac{1}{k-2+p}<n+\sum_{k=1}^{n}\frac{1}{k-2+2}=n+h_n,\ \text{where } h_n:=\sum_{k=1}^{n}\frac{1}{k}.$$

Hence $\frac{1}{x_n}<p+n+h_n, n\in\mathbb{N}$ and, since* $h_n<\sqrt{n}+1,\ n\in\mathbb{N}$
then $\frac{1}{x_n}<p+n+\sqrt{n}+1 \iff \frac{1}{n+\sqrt{n}+p+1}<x_n<\frac{1}{n+p}$.
*(for $n\geq 3$ we have $n^2>2n+1 \implies n^2>2\sqrt{n(n+1)} \iff$
$\sqrt{n+1}+\sqrt{n}<n+1 \iff \frac{1}{n+1}<\frac{1}{\sqrt{n+1}+\sqrt{n}} \iff$
$h_{n+1}-h_n<\sqrt{n+1}-\sqrt{n}$ and $h_1=1<\sqrt{1}+1, h_2<\sqrt{2}+1$,
$h_3<\sqrt{3}+1$)

8. Inequalities and max,min problems.

Comparison of numerical expressions.
Problem 8.1(Met. Rec.)
a) $31^{11}<32^{11}=2^{55}<2^{56}=16^{14}<17^{14}$;
b) $513^{18}>512^{18}=2^{9\cdot 18}=2^{162}>2^{161}=2^{7\cdot 23}=128^{23}>127^{23}$;
c) Particular case of inequality $n^{\frac{1}{n}}>m^{\frac{1}{m}}$ if $3\leq n<m$.

Suffice to prove $n^{\frac{1}{n}} \downarrow \mathbb{N} \setminus \{1,2\}$.

d) Answer: $\tan 34^\circ > \dfrac{2}{3}$

Since $\tan 34^\circ = \dfrac{1/\sqrt{3}+\tan 4^\circ}{1-1/\sqrt{3}\tan 4^\circ}$ and $\dfrac{1/\sqrt{3}+t}{1-1/\sqrt{3}\cdot t} \uparrow \left(0,\sqrt{3}\right)$,

$\tan 4^\circ = \tan\dfrac{\pi}{45} > \dfrac{\pi}{45}$ then

$\tan 34^\circ = \dfrac{1/\sqrt{3}+\tan(\pi/45)}{1-1/\sqrt{3}\tan(\pi/45)} > \dfrac{1/\sqrt{3}+\pi/45}{1-1/\sqrt{3}\cdot\pi/45}$ and

$\dfrac{1/\sqrt{3}+\pi/45}{1-1/\sqrt{3}\cdot\pi/45} > \dfrac{2}{3} \iff \sqrt{3}+\dfrac{\pi}{15} > 2-\dfrac{2\pi}{45\sqrt{3}} \iff$

$\dfrac{\pi}{15}+\dfrac{2\pi}{45\sqrt{3}} > 2-\sqrt{3} \iff \pi\left(3\sqrt{3}+2\right) > 45\left(2\sqrt{3}-3\right)$.

We have $3\left(3\sqrt{3}+2\right) > 45\left(2\sqrt{3}-3\right) \iff 3\sqrt{3}+2 > 15\left(2\sqrt{3}-3\right) \iff$

$\dfrac{3\sqrt{3}+2}{2\sqrt{3}-3} > 15 \iff \left(3\sqrt{3}+2\right)\left(2\sqrt{3}+3\right) > 45 \iff 13\sqrt{3}+24 > 45 \iff$

$13\sqrt{3} > 21 \iff 169\cdot 3 > 441 \iff 507 > 441.$

e) Since $1 \in (0,\pi/2)$ and $1 > \pi/4$ then $\sin 1 > \cos 1$.

Also since $\sin\dfrac{1}{2} < \dfrac{1}{2}$ then $\cos 1 = 1-2\sin^2\dfrac{1}{2} > 1-2\cdot\left(\dfrac{1}{2}\right)^2 = \dfrac{1}{2}$.

Thus, $\sin 1 > \dfrac{1}{2}$.(Or, since $1 > \pi/6$ then $\sin 1 > \sin\pi/6 = \dfrac{1}{2}$.

From the other hand $\log_3\sqrt{2} = \dfrac{1}{2}\log_3 2 < \dfrac{1}{2}$.

Hence, $\sin 1 > \dfrac{1}{2} > \log_3\sqrt{2}$.

f) Solution1.

Since $n-1 \geq 2$ then $\log_{n-1} n > \log_n(n+1) \iff 1 > \log_n(n-1)\cdot\log_n(n+1) \iff$

$1 > \log_n n\left(1-\dfrac{1}{n}\right)\cdot\log_n n\left(1+\dfrac{1}{n}\right) \iff 1 > \left(1+\log_n\left(1-\dfrac{1}{n}\right)\right)\left(1+\log_n\left(1+\dfrac{1}{n}\right)\right) \iff$

$1 > 1+\log_n\left(1-\dfrac{1}{n}\right)+\log_n\left(1+\dfrac{1}{n}\right)+\log_n\left(1-\dfrac{1}{n}\right)\log_n\left(1+\dfrac{1}{n}\right) \iff$

$0 > \log_n\left(1-\dfrac{1}{n^2}\right)+\log_n\left(1-\dfrac{1}{n}\right)\log_n\left(1+\dfrac{1}{n}\right)$

where latter inequality holds because

$\log_n\left(1-\dfrac{1}{n^2}\right) < 0, \log_n\left(1-\dfrac{1}{n}\right) < 0$ and $\log_n\left(1+\dfrac{1}{n}\right) > 0$.

Solution2.

By **(2AGM)** $\log_n(n-1)\cdot\log_n(n+1) < \left(\dfrac{\log_n(n-1)+\log_n(n+1)}{2}\right)^2 \iff$

$\log_n(n-1)\cdot\log_n(n+1) < \left(\dfrac{\log_n\left(n^2-1\right)}{2}\right)^2$ and $\dfrac{\log_n\left(n^2-1\right)}{2} < \dfrac{\log_n n^2}{2} =$

$\dfrac{2\log_n n}{2} = 1$ then $\log_n(n-1)\cdot\log_n(n+1) < 1 \iff \log_{n-1} n > \log_n(n+1)$.

h) Using Math Induction we will prove that for any natural n holds inequality $n! > \left(\dfrac{n}{3}\right)^n$.
Note that for $n = 1$ this inequality obviously holds.
For any $n \in \mathbb{N}$ assuming $n! > \left(\dfrac{n}{3}\right)^n$ we obtain $(n+1)! > (n+1)\left(\dfrac{n}{3}\right)^n$ and

$$(n+1)\left(\frac{n}{3}\right)^n > \left(\frac{n+1}{3}\right)^{n+1} \iff \frac{n^n}{3^n} > \frac{(n+1)^n}{3^{n+1}} \iff$$

$$3 > \left(1+\frac{1}{n}\right)^n \Longleftarrow 3 > e > \left(1+\frac{1}{n}\right)^n.$$

Applying inequality $n! > \left(\dfrac{n}{3}\right)^n$ for $n = 300$ we obtain

$$300! > \left(\frac{300}{3}\right)^{300} = 100^{300}.$$

Remark.
Another way to prove inequality $3 > \left(1+\dfrac{1}{n}\right)^n$ without reference to e.
We will prove (using Math Induction) one useful inequality, namely:
For any positive real α and any natural n such that $n\alpha \le 1$ holds inequality
(1) $(1+\alpha)^n < 1 + n\alpha + n^2\alpha^2$.
Proof.
For $n = 1$ inequality obviously holds.
Let $n \in \mathbb{N}$ be any such that $(n+1)\alpha \le 1$. Then $n\alpha < 1$ and assuming $(1+\alpha)^n < 1 + n\alpha + n^2\alpha^2$ we obtain
$(1+\alpha)^{n+1} < \left(1 + n\alpha + n^2\alpha^2\right)(1+\alpha) = n^2\alpha^3 + \left(n^2+n\right)\alpha^2 + (n+1)\alpha + 1$.
Since $n\alpha < 1$ we have $n^2\alpha^3 = n\alpha\cdot n\alpha^2 < n\alpha^2$ and, therefore,
$n^2\alpha^3 + \left(n^2+n\right)\alpha^2 < \left(n^2+2n\right)\alpha^2 < (n+1)^2\alpha^2$.
Hence, $(1+\alpha)^{n+1} < 1 + (n+1)\alpha + (n+1)^2\alpha^2$.
Applying inequality **(1)** to $\alpha = \dfrac{1}{n}$ we obtain $\left(1+\dfrac{1}{n}\right)^n < 3$.

g). Easy to see that $(n!)^2 = n^n$ for $n = 1, 2$ and for $n = 3$ we have $(3!)^2 = 36 > 3^3$. We will prove that $(n!)^2 > n^n$ for any natural $n \ge 3$ using Math Induction in form of Multiplicative Reduction, that is suffice to prove inequality

$$\frac{((n+1)!)^2}{(n!)^2} > \frac{(n+1)^{n+1}}{n^n} \iff n+1 > \left(1+\frac{1}{n}\right)^n.$$

Latter inequality immediately follows from $3 > \left(1+\dfrac{1}{n}\right)^n$,or can be proved independently with usage of Multiplicative Reduction, namely we have

$$\frac{n+2}{n+1}=1+\frac{1}{n+1}>\frac{\left(1+\dfrac{1}{n+1}\right)^{n+1}}{\left(1+\dfrac{1}{n}\right)^{n}} \iff$$

$$\left(1+\frac{1}{n}\right)^{n}>\left(1+\frac{1}{n+1}\right)^{n} \iff 1+\frac{1}{n}>1+\frac{1}{n+1}.$$

i). Let $a_n:=\sqrt{2+\sqrt{3+\sqrt{2+\ldots}}}$ and $b_n:=\sqrt{3+\sqrt{2+\sqrt{3+\ldots}}}$
(each use n square root symbols)
We have system of recurrences:

$$\begin{cases} a_{n+1}=\sqrt{2+b_n} \\ b_{n+1}=\sqrt{3+a_n} \end{cases},n\in\mathbb{N} \text{ where } a_1=\sqrt{2},b_1=\sqrt{3}.$$

Note that $a_1<b_1$ and $b_1-a_1<1$ and we will prove using Mat. Induction more stronger inequality $0<b_n-a_n<1$ for any natural n.
1. Base of induction:
$0<\sqrt{3}-\sqrt{2}<1.$
2. Step of Induction.
For any $n\in\mathbb{N}$ in supposition that $0<b_n-a_n<1$ we we have
$b_{n+1}-a_{n+1}=\sqrt{3+a_n}-\sqrt{2+b_n}=$
$\dfrac{1-(b_n-a_n)}{\sqrt{2+b_n}+\sqrt{3+a_n}}<\dfrac{1}{\sqrt{2}+\sqrt{3}}<1$ and $b_{n+1}-a_{n+1}>0.$

Proving inequalities

Problem 8.2
Since $a+b+c=0$ then due to symmetry of inequality we can assume that $sign\,(a)=sign\,(b)$. Then $|c|=|a+b|=|a|+|\,b|$ and

$$|a\cdot b\cdot c|=|a|\cdot|b|\cdot(|a|+|\,b|)\le\left(\frac{|a|+|\,b|}{2}\right)^2(|a|+|\,b|)=\frac{(|a|+|\,b|)^3}{4}=$$

$$\frac{|c|^3}{4}=\frac{1}{4}\max\left\{|a|^3,|b|^3,|c|^3\right\} \text{ because } |a|,|\,b|\le|a|+|\,b|=|c|.$$

Problem 8.3(Met. Rec.).

We have

$$(1-x)(1-y)(1-z)-\frac{1}{2}=\left(\frac{1}{2}-(x+y+z)\right)+xy(1-z)+z(y+x)z\ge 0.$$

Problem 8.4(Problem 6 from 6-th CGMO,2-nd day,2007).
Due to symmetry with respect to b and c we can assume that $b\ge c$ and denoting $x:=\sqrt{b}+\sqrt{c},y:=\sqrt{b}-\sqrt{c}$ we obtain $x\ge y\ge 0,x+y\le 1,$
$b-c=xy,b+c=\dfrac{x^2+y^2}{2},a=1-\dfrac{x^2+y^2}{2},$
and original inequality becomes

(1) $$\sqrt{1-\frac{x^2+y^2}{2}+\frac{x^2y^2}{4}}+x\le\sqrt{3},$$

where $x \in \left[\frac{1}{2}, 1\right]$ and $y \in [0, x]$.

Since $\max\limits_{y\in[0,x]}\left(-\frac{y^2}{2}+\frac{x^2y^2}{4}\right) = \max\limits_{y\in[0,x]}\left(-\frac{y^2\left(2-x^2\right)}{4}\right) = 0$ then

(1) $\iff \max\limits_{y\in[0,x]}\sqrt{1-\frac{x^2+y^2}{2}+\frac{x^2y^2}{4}}+x \le \sqrt{3} \iff$

$\sqrt{1-\frac{x^2}{2}+\max\limits_{y\in[0,x]}\left(-\frac{y^2}{2}+\frac{x^2y^2}{4}\right)}+x \le \sqrt{3} \iff \sqrt{1-\frac{x^2}{2}}+x \le \sqrt{3},$

where latter inequality holds because by Cauchy Inequality

$\left(\sqrt{1-\frac{x^2}{2}}+x\right)^2 = \left(1\cdot\sqrt{1-\frac{x^2}{2}}+\sqrt{2}\cdot\frac{x}{\sqrt{2}}\right)^2 \le$

$\left(1^2+\left(\sqrt{2}\right)^2\right)\left(\left(\sqrt{1-\frac{x^2}{2}}\right)^2+\left(\frac{x}{\sqrt{2}}\right)^2\right) = 3\cdot\left(1-\frac{x^2}{2}+\frac{x^2}{2}\right) = 3.$

Since in (1) equality occurs iff $y = 0$ and $\sqrt{1-\frac{x^2}{2}} \div \frac{x}{\sqrt{2}} = 1 \div \sqrt{2} \iff$

$y = 0$ and $x = \frac{2}{\sqrt{3}}$ then original inequality becomes equality iff

$a = b = c = \frac{1}{3}.$

Remark.

This inequality is sharp variant of inequality $\sqrt{a}+\sqrt{b}+\sqrt{c} \le \sqrt{3}.$

Problem 8.5(Met. Rec.)

We have $\sum\limits_{i=1}^{n}\frac{a_i - a_{i+2}}{a_{i+1}+a_{i+2}} = \sum\limits_{i=1}^{n}\frac{a_i + a_{i+1} - (a_{i+1}+a_{i+2})}{a_{i+1}+a_{i+2}} = \sum\limits_{i=1}^{n}\left(\frac{a_i+a_{i+1}}{a_{i+1}+a_{i+2}}-1\right) =$

$\sum\limits_{i=1}^{n}\frac{a_i+a_{i+1}}{a_{i+1}+a_{i+2}} - n \ge 0$ because by AM-GM Inequality

$$\sum_{i=1}^{n}\frac{a_i+a_{i+1}}{a_{i+1}+a_{i+2}} \ge n\sqrt[n]{\prod_{i=1}^{n}\frac{a_i+a_{i+1}}{a_{i+1}+a_{i+2}}} = n.$$

Problem 8.6(Met. Rec.)

Let $S := \sum\limits_{cyc}\frac{a^3}{a^2+ab+b^2}.$

Since $\frac{a^3}{a^2+ab+b^2} = \frac{a^3-b^3+b^3}{a^2+ab+b^2} = a - b + \frac{b^3}{a^2+ab+b^2}$

then $S = \sum\limits_{cyc}\left(a-b+\frac{b^3}{a^2+ab+b^2}\right) = \sum\limits_{cyc}\frac{b^3}{a^2+ab+b^2}$

and, therefore,

$2S = \sum\limits_{cyc}\frac{a^3+b^3}{a^2+ab+b^2} = \sum\limits_{cyc}(a+b)\frac{a^2-ab+b^2}{a^2+ab+b^2} \ge \sum\limits_{cyc}(a+b)\cdot\frac{1}{3} = \frac{2(a+b+c)}{3}$

because $\frac{a^2-ab+b^2}{a^2+ab+b^2} \ge \frac{1}{3} \iff (a-b)^2 \ge 0.$

Problem 8.7(Met. Rec)

By Chebishev's Inequality we have $a^5+b^5+c^5 \geq \left(a^2+b^2+c^2\right)\cdot\dfrac{a^3+b^3+c^3}{3}$ and $a^2+b^2+c^2 \geq ab+bc+ca$, $\dfrac{a^3+b^3+c^3}{3} \geq abc$.

Problem 8.8(Met. Rec).
Solution 1.
Applying inequality $(x+y+z)^2 \leq 3\left(x^2+y^2+z^2\right)$, $x,y,z \in \mathbb{R}$ to
$(x,y,z) = \left(\sqrt{4a+1}, \sqrt{4b+1}, \sqrt{4c+1}\right)$
we obtain
$\left(\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\right)^2 \leq 3\,(4a+1+4b+1+4c+1) = 21.$

Solution 2.
First note that
$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \leq \sqrt{21} \iff$
$\left(\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\right)^2 \leq 21 \iff$
$4a+1+4b+1+4c+1+\sum\limits_{cyc} 2\sqrt{(4a+1)(4b+1)} \leq 21 \iff$
$\sum\limits_{cyc} 2\sqrt{(4a+1)(4b+1)} \leq 14.$
Since, by 2AM-GM Inequality
$2\sqrt{(4a+1)(4b+1)} \leq 4a+1+4b+1 = 6-4c$ then
$\sum\limits_{cyc} 2\sqrt{(4a+1)(4b+1)} \leq \sum\limits_{cyc} (6-4c) = 18-4 = 14.$

Solution 3.
Let $t>0$ be some undetermined real parameter.
Then using 2AM-GM inequality we obtain
$\sum\limits_{cyc} \sqrt{4a+1} = \dfrac{1}{\sqrt{t}}\sum\limits_{cyc}\sqrt{t\,(4a+1)} \leq \dfrac{1}{2\sqrt{t}}\sum\limits_{cyc}(t+4a+1) = \dfrac{7+3t}{2\sqrt{t}}.$
So, for any $t>0$ holds inequality

(1) $\quad \sum\limits_{cyc}\sqrt{4a+1} \leq \dfrac{7+3t}{2\sqrt{t}}$

To reach upper bound $\dfrac{7+3t}{2\sqrt{t}}$ we should claim $t = 4a+1 = 4b+1 = 4c+1$.

Since $a+b+c=1$ we obtain $3t = 4\,(a+b+c)+3 = 7 \iff t = \dfrac{7}{3}$.

In particular for $t = \dfrac{7}{3}$ inequality **(1)** becomes

$$\sum_{cyc}\sqrt{4a+1} \leq \frac{7+3\cdot\dfrac{7}{3}}{2\sqrt{\dfrac{7}{3}}} = \sqrt{21}.$$

Another ending.

We have $\sum\limits_{cyc}\sqrt{4a+1} \leq \dfrac{7+3t}{2\sqrt{t}} \iff \sum\limits_{cyc}\sqrt{4a+1} \leq \min\limits_{t>0}\dfrac{7+3t}{2\sqrt{t}}$

and $\dfrac{7+3t}{2\sqrt{t}} = \dfrac{1}{2}\left(\dfrac{7}{\sqrt{t}} + 3\sqrt{t}\right) \geq \sqrt{\dfrac{7}{\sqrt{t}} \cdot 3\sqrt{t}} = 21$ with equality iff
$\dfrac{7}{\sqrt{t}} = 3\sqrt{t} \iff t = \dfrac{7}{3}$.

Problem 8.9(Met. Rec).
Since $(x_1 + x_2 + ... + x_n + 1)^2 \geq 4(x_1 + x_2 + ... + x_n) \iff$
$(x_1 + x_2 + ... + x_n - 1)^2 \geq 0$
and $x_i \in [0,1] \implies x_i \geq x_i^2, i = 1, 2, ..., n$ we have
$(x_1 + x_2 + ... + x_n + 1)^2 \geq 4(x_1 + x_2 + ... + x_n) \geq 4\left(x_1^2 + x_2^2 + ... + x_n^2\right)$.

Problem 8.10(Met. Rec).
Solution1.
First note that $x^3 z + y^3 x + z^3 y \geq xyz(x + y + z) \iff$
$\dfrac{x^2}{y} + \dfrac{y^2}{z} + \dfrac{z^2}{x} \geq x + y + z$.
Applying Cauchy Inequality to triples $\left(\dfrac{x}{\sqrt{y}}, \dfrac{y}{\sqrt{z}}, \dfrac{z}{\sqrt{x}}\right), (\sqrt{y}, \sqrt{z}, \sqrt{x})$
we obtain
$\left(\dfrac{x^2}{y} + \dfrac{y^2}{z} + \dfrac{z^2}{x}\right)(y + z + x) \geq \left(\dfrac{x}{\sqrt{y}} \cdot \sqrt{y} + \dfrac{y}{\sqrt{z}} \cdot \sqrt{z} + \dfrac{z}{\sqrt{x}} \cdot \sqrt{x}\right)^2 = (x + y + z)^2 \iff$
$\dfrac{x^2}{y} + \dfrac{y^2}{z} + \dfrac{z^2}{x} \geq x + y + z$.

Solution2.
Note that for any real a and real $b > 0$ holds inequality $\dfrac{a^2}{b} \geq 2a - b$ ($\iff$ $(a - b)^2 \geq 0$).
Then $\dfrac{x^2}{y} + \dfrac{y^2}{z} + \dfrac{z^2}{x} \geq 2y - x + 2y - z + 2z - x = x + y + z$.

Problem 8.11(Met. Rec).
Noting that $\dfrac{x_1 y_1 + x_2 y_1}{x_1 y_1 + x_1 y_2} = \dfrac{y_1(x_1 + x_2)}{x_1(y_1 + y_2)}, \dfrac{x_2 y_2 + x_1 y_2}{x_2 y_2 + x_2 y_1} = \dfrac{y_2(x_1 + x_2)}{x_2(y_1 + y_2)}$
and using Weighted AM-GM Inequality we obtain that

$$\left(\frac{x_1 y_1 + x_2 y_1}{x_1 y_1 + x_1 y_2}\right)^{x_1} \left(\frac{x_2 y_2 + x_1 y_2}{x_2 y_2 + x_2 y_1}\right)^{x_2} \leq \left(\frac{\dfrac{y_1(x_1 + x_2)}{x_1(y_1 + y_2)} \cdot x_1 + \dfrac{y_2(x_1 + x_2)}{x_2(y_1 + y_2)} \cdot x_2}{x_1 + x_2}\right)^{x_1 + x_2} = 1$$

for any real positive x_1, x_2, y_1, y_2.
Thus, in fact, all solutions of inequality of the problem are solution of equation
$$\left(\frac{x_1 y_1 + x_2 y_1}{x_1 y_1 + x_1 y_2}\right)^{x_1} \left(\frac{x_2 y_2 + x_1 y_2}{x_2 y_2 + x_2 y_1}\right)^{x_2} = 1.$$
By condition of equality in weighted AM-GM Inequality we obtain

$$\frac{x_1y_1+x_2y_1}{x_1y_1+x_1y_2}=\frac{x_2y_2+x_1y_2}{x_2y_2+x_2y_1} \iff \frac{y_1(x_1+x_2)}{x_1(y_1+y_2)}=\frac{y_2(x_1+x_2)}{x_2(y_1+y_2)} \iff \frac{y_1}{x_1}=\frac{y_2}{x_2}.$$

Thus all solution represented by quads $(x_1,x_2,y_1,y_2)=(x_1,x_2,tx_1,tx_2)$,where $x_1,x_2,t\in(0,\infty)$.

Another variant of previous solution..

Denoting $u_i:=\dfrac{x_i}{x_1+x_2}$ and $v_i:=\dfrac{y_i}{y_1+y_2} i=1,2$

we obtain that $u_1+u_2=v_1+v_2=1$ and

$\dfrac{x_1y_1+x_2y_1}{x_1y_1+x_1y_2}=\dfrac{y_1(x_1+x_2)}{x_1(y_1+y_2)}=\dfrac{v_1}{u_1}, \dfrac{x_2y_2+x_1y_2}{x_2y_2+x_2y_1}=\dfrac{v_2}{u_2}$. Then inequality

$\left(\dfrac{x_1y_1+x_2y_1}{x_1y_1+x_1y_2}\right)^{x_1}\left(\dfrac{x_2y_2+x_1y_2}{x_2y_2+x_2y_1}\right)^{x_2}\geq 1$ becomes $\left(\dfrac{v_1}{u_1}\right)^{u_1}\left(\dfrac{v_2}{u_2}\right)^{u_2}\geq 1.$

Since by weighted AM-GM Inequality

$$\left(\frac{v_1}{u_1}\right)^{u_1}\left(\frac{v_2}{u_2}\right)^{u_2}\leq\frac{v_1}{u_1}\cdot u_1+\frac{v_2}{u_2}\cdot u_2=v_1+v_2=1$$

then $\left(\dfrac{v_1}{u_1}\right)^{u_1}\left(\dfrac{v_2}{u_2}\right)^{u_2}=1$ and it is possible iff $\dfrac{v_1}{u_1}=\dfrac{v_2}{u_2}$

(condition of equality in AM-GM Inequality).

That is, $(v_1,v_2)=t(u_1,u_2) \iff (y_1,y_2)=t(x_1,x_2), t\in(0,\infty)$.

And one more proof of inequality $\left(\dfrac{v_1}{u_1}\right)^{u_1}\left(\dfrac{v_2}{u_2}\right)^{u_2}\leq 1$

(without weighted AM-GM Inequality).

First note that

$$\left(\frac{v_1}{u_1}\right)^{u_1}\left(\frac{v_2}{u_2}\right)^{u_2}\leq 1 \iff \left(\frac{v_1}{u_1}\right)^{u_1}\left(\frac{v_2}{u_2}\right)^{1-u_1}\leq 1 \iff \left(\frac{v_1u_2}{u_1v_2}\right)^{u_1}\leq\frac{u_2}{v_2}.$$

Applying Bernoulli-2 Inequality $(1+t)^{\alpha}\leq 1+t\alpha, t>-1,\ \alpha\in(0,]$

to $t=\dfrac{v_1u_2}{u_1v_2}-1$ and $\alpha=u_1$

we obtain

$$\left(\frac{v_1u_2}{u_1v_2}\right)^{u_1}\leq 1+\left(\frac{v_1u_2}{u_1v_2}-1\right)u_1=1-u_1+\frac{v_1u_2}{v_2}=u_2+\frac{v_1u_2}{v_2}=\frac{u_2(v_1+v_2)}{v_2}=\frac{u_2}{v_2}.$$

Problem 8.12(Met. Rec).

Let $S_n:=\sum\limits_{k=1}^{n}\sqrt{2k-2\sqrt{k(k-1)}}$and $b_n:=\sqrt{n(n+1)}$.

First note that $S_1=\sqrt{2}=b_1$.

Also note that $S_2=\sqrt{2}+\sqrt{4-2\sqrt{2}}>\sqrt{6}=b_2$.

Indeed, $\sqrt{2}+\sqrt{4-2\sqrt{2}}>\sqrt{6} \iff 1+\sqrt{2-\sqrt{2}}>\sqrt{3} \iff$
$2-\sqrt{2}>\left(\sqrt{3}-1\right)^2 \iff 2-\sqrt{2}>4-2\sqrt{3} \iff 2\sqrt{3}>2+\sqrt{2} \iff$
$\sqrt{6}>\sqrt{2}+1 \iff 6>3+2\sqrt{2} \iff 3>2\sqrt{2} \iff 9>8$.

And we will prove that for any $n\geq 2$ holds $S_n-S_{n-1}>b_n-b_{n-1}$.

Indeed, $\sqrt{2n-2\sqrt{n(n-1)}} > \sqrt{n(n+1)} - \sqrt{n(n-1)} \iff$
$2n-2\sqrt{n(n-1)} > 2n^2-2n\sqrt{n^2-1} \iff \dfrac{n^2-(n^2-n)}{n+\sqrt{n(n-1)}} > \dfrac{n^4-(n^4-n^2)}{n^2+n\sqrt{n^2-1}} \iff$
$\dfrac{n}{n+\sqrt{n(n-1)}} > \dfrac{n^2}{n^2+n\sqrt{n^2-1}} \iff \dfrac{1}{n+\sqrt{n(n-1)}} > \dfrac{1}{n+\sqrt{n^2-1}} \iff$
$n+\sqrt{n^2-1} > n+\sqrt{n(n-1)} \iff \sqrt{n+1} > \sqrt{n}.$
Since $S_2 > b_2$ and for any $n \geq 3,$ assuming that $S_{n-1} \geq b_{n-1}$ we obtain
$S_n = (S_n - S_{n-1}) + S_{n-1} \geq (b_n - b_{n-1}) + b_{n-1} = b_n$ then by Math Induction
we have $S_n > b_n$ for any $n \geq 2$ and, therefore, $S_n \geq b_n$ for any $n \geq 1$.

Problem 8.13(Met. Rec).

Let $s_k := \sum\limits_{i=1}^{k} a_i, k = 1,2,...,n$ and $f(x) := \sqrt{1-x^2}$ then
$F_n := \sum\limits_{k=1}^{n} a_k \sqrt{1-\left(\sum\limits_{i=1}^{k} a_i\right)^2} = \sum\limits_{k=1}^{n} (s_k - s_{k-1}) \sqrt{1-s_k^2} = \sum\limits_{k=1}^{n} (s_k - s_{k-1}) f(s_k)$
is Riemann sum for function $f(x) = \sqrt{1-x^2}$ and partition
$0 = s_0 < s_1 < s_2 < ... < s_n = 1$ of the segment $[0,1]$.
Since, $f(x) \downarrow [0,1]$ then
$\sum\limits_{k=1}^{n} (s_k - s_{k-1}) f(s_k) < \int\limits_0^1 \sqrt{1-x^2}dx = [x = \sin t; dx = \cos t \cdot dt] =$
$\int\limits_0^{\pi/2} \cos^2 t dt = \dfrac{1}{2}\int\limits_0^{\pi/2} (1+\cos 2t)\, dt = \dfrac{1}{2}\left(t + \dfrac{\sin 2t}{2}\right)_0^{\pi/2} = \dfrac{\pi}{4} < \dfrac{4}{5}$
because $5\pi < 16$.

Problem 8.14(Met. Rec).

If $n = 2$ then $a_1a_2 + a_2a_1 \leq \dfrac{(a_1+a_2)^2}{2} \iff (a_1-a_2)^2 \geq 0;$
If $n = 3$ then ${}_1a_2 + a_2a_3 + a_3a_1 \leq \dfrac{(a_1+a_2+a_3)^2}{3} \iff$
$(a_1-a_2)^2 + (a_2-a_3)^2 + (a_3-a_1)^2 \geq 0;$
Let $n > 3$.
Due to cyclic symmetry of the inequality we can suppose that
$\min\{a_1, a_2, ..., a_n\} = a_1$.
Then we have
$a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1 \leq a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_{n-3}a_n \leq$
$\sum\limits_{i-odd}^{n} a_i \cdot \sum\limits_{j-even}^{n} a_j \leq \dfrac{(a_1+a_2+...+a_n)^2}{4}$
because in the each term $a_ia_{i+1}, i = 1,2,...,n-1$
and $a_{n-3}a_n$ one factor has odd index, other has even index.
Equality occurs, for example, if $a_1 = a_2$ and all other $a_i = 0,\ i = 3,...,n$.

Problem 8.15.(Met. Rec).Original setting.

Let $S = S(a_1, a_2, ..., a_n) := \sum_{1\le i<j\le n}^{n} |a_i - a_j|$. Since $|a_i - a_i| = 0, i \in \{1, 2, ..., n\}$

then $\sum_{i=1}^{n}\sum_{j=1}^{n} |a_i - a_j| = 2S \iff S = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |a_i - a_j|$ and, therefore,

$S(a_1, a_2, ..., a_n)$ is independent from permutations of $(a_1, a_2, ..., a_n)$

and for any real a holds

$S(a_1 + a, a_2 + a, ..., a_n + a) = S(a_1, a_2, ..., a_n)$.

Thus, $\max\{S(a_1, a_2, ..., a_n) \mid a_1, a_2, ..., a_n \in \mathbb{R} \text{ and } |a_i - a_j| \le 2, i, j \in \{1, 2, ..., n\}\} =$
$\max\{S(a_1, a_2, ..., a_n) \mid 0 \le a_1 \le a_2 \le ... \le a_n \le 2\}$.

Since $0 \le a_1 \le a_2 \le ... \le a_n \le 2$ then $S = \sum_{1\le i<j\le n}^{n} |a_i - a_j| = \sum_{1\le i<j\le n}^{n} (a_j - a_i) =$

$\sum_{j=2}^{n}\sum_{i=1}^{j-1} a_j - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i = \sum_{j=2}^{n} (j-1) a_j - \sum_{i=1}^{n-1} (n-i) a_i =$

$(n-1) a_n + \sum_{j=2}^{n-1} (j-1) a_j - (n-1) a_1 - \sum_{i=2}^{n-1} (n-i) a_i =$

$(n-1)(a_n - a_1) + \sum_{i=2}^{n-1} (2i - 1 - n) a_i$.

If $n = 2$ then $\sum_{i=2}^{n-1} (2i - 1 - n) a_i = 0$ and

$S = \frac{1}{2}(2-1)(a_n - a_1) \le \frac{1}{2}(2-0) = 1$.

Let $n \ge 3$.Since $a_n - a_1 \le 2 - 0 = 2$ and $2i - 1 - n \ge 0 \iff i \ge \left[\frac{n}{2}\right] + 1$,

$n + 1 - 2i \ge 1 \iff i \le \left[\frac{n}{2}\right]$ then

$S = (n-1)(a_n - a_1) + \sum_{i=\left[\frac{n}{2}\right]+1}^{n-1} (2i - 1 - n) a_i - \sum_{i=2}^{\left[\frac{n}{2}\right]} (n + 1 - 2i) a_i \le$

$2(n-1) + 2\sum_{i=\left[\frac{n}{2}\right]+1}^{n-1} (2i - 1 - n)$.

Consider now two cases:

1. If $n = 2m$ then $\sum_{i=\left[\frac{n}{2}\right]+1}^{n-1} (2i - 1 - n) = \sum_{i=m+1}^{2m-1} (2i - 1 - 2m) = \sum_{i=m+1}^{2m-1} (2(i-m) - 1) =$

$\sum_{i=1}^{m-1} (2i - 1) = (m-1)^2$ and, therefore, $S \le 2(2m-1) + 2(m-1)^2 = 2m^2$.

2. If $n = 2m-1$ then $\sum_{i=\left[\frac{n}{2}\right]+1}^{n-1} (2i - 1 - n) = \sum_{i=m}^{2m-2} (2i - 2m) = 2\sum_{i=m+1}^{2m-2} (i - m) =$

$2\sum_{i=1}^{m-2} i = (m-2)(m-1)$ and, therefore,

$S \le 2(2m-2) + 2(m-2)(m-1) = 2m(m-1)$.

Thus, $S \le \begin{cases} 2m^2 & \text{if } n = 2m \\ 2m(m-1) & \text{if } n = 2m-2 \end{cases}$.

Since, $\left[\dfrac{n^2}{2}\right] = \begin{cases} 2m^2 & \text{if } n = 2m \\ 2m(m-1) & \text{if } n = 2m-1 \end{cases}$ then $S \leq \left[\dfrac{n^2}{2}\right]$.

Noting that $S(a_1, a_2, ..., a_n) = \left[\dfrac{n^2}{2}\right]$ for

$a_1 = a_2 = ... = a_{\left[\frac{n}{2}\right]} = 0, a_{\left[\frac{n}{2}\right]+1} = ... = a_n = 2$

we conclude that $\max S(a_1, a_2, ..., a_n) = \left[\dfrac{n^2}{2}\right]$.

Hence, $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} |a_i - a_j| \leq 2\left[\dfrac{n^2}{2}\right] \leq n^2$.

Problem 8.16 (MR S97as modification)

Let $t := (x_1 x_2 ... x_n)^{\frac{2}{n}}$ then by AM-GM Inequality

$1 = \dfrac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n} \implies t \leq 1.$

Since

$$x_1^2 + x_2^2 + ... + x_n^2 = (x_1 + x_2 + ... + x_n)^2 - 2\sum_{1 \leq i < j \leq n} x_i x_j = n^2 - 2\sum_{1 \leq i < j \leq n} x_i x_j$$

and by AM-GM Inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq \binom{n}{2}\left(\prod_{1 \leq i < j \leq n} x_i x_j\right)^{\frac{1}{\binom{n}{2}}} = \binom{n}{2}\left(\prod_{k=1}^{n} x_k^{n-1}\right)^{\frac{2}{n(n-1)}} =$$

$\dfrac{n(n-1)}{2} \prod\limits_{k=1}^{n} x_k^{\frac{2}{n}} = \dfrac{n(n-1)}{2}$ then $x_1^2 + x_2^2 + ... + x_n^2 \leq n^2 - n(n-1)t$ and

$x_1^2 x_2^2 ... x_n^2 \left(x_1^2 + x_2^2 + ... + x_n^2\right) \leq t^n \left(n^2 - n(n-1)t\right)$,

Thus suffices to prove

$t^n \left(n^2 - n(n-1)t\right) \leq n \iff nt^n - (n-1)t^{n+1} \leq 1 \iff$

$(n-1)t^{n+1} - nt^n + 1 \geq 0$ for any $t \in [0, 1]$.

Latter inequality holds because

$(n-1)t^{n+1} - nt^n + 1 = (n-1)t^n(t-1) - (t^n - 1) =$

$(t-1)\left((n-1)t^n - \left(1 + t + ... + t^{n-1}\right)\right) = (1-t)\left(1 + t + ... + t^{n-1} - (n-1)t^n\right) \geq$

$(1-t)\left(nt^{n-1} - (n-1)t^n\right) = t^{n-1}(1-t)(1 + (n-1)(1-t)) \geq 0.$

★ **Problem 8.17 (W6 J Wildt IMO, 2014)**

Let $S(\mathbf{x}_{\mathbb{N}}) := \sum\limits_{n=1}^{\infty} \dfrac{x_n^3}{x_n + 4x_{n+1}}$ if series converges and $S_f(x_{\mathbb{N}}) = \infty$ if it diverges.

Let $\widetilde{D}_1 = \{x_{\mathbb{N}} \mid x_{\mathbb{N}} \in D_1 \text{ and } S(x_{\mathbb{N}}) \neq \infty\}$ .Since $\widetilde{D}_1$ isn't empty (because for for instance if $x_n = q^{n-1}, n \in \mathbb{N}$, where $q \in (0, 1)$, we have

$\sum\limits_{n=1}^{\infty} \dfrac{x_n^3}{x_n + 4x_{n+1}} = \sum\limits_{n=1}^{\infty} \dfrac{q^{3(n-1)}}{q^{n-1} + 4q^n} = \sum\limits_{n=1}^{\infty} \dfrac{q^{2(n-1)}}{1 + 4q} = \dfrac{1}{(1+4q)(1-q^2)}$

then $\inf\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in D_1\} = \inf\left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$.
Let $S := \inf\left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$. For any $\mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1$ we have

$$S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{1+4x_2} + \sum_{n=2}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$$
$$\frac{1}{1+4x_2} + x_2^2 \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} = \frac{1}{1+4x_2} + x_2^2 \cdot S(\mathbf{y}_{\mathbb{N}}),$$

where $y_n := \dfrac{x_{n+1}}{x_2}, n \in \mathbb{N}$.
Since $\mathbf{y}_{\mathbb{N}} \in \widetilde{D}_1$ ($1 = y_1 > y_2 > .... > y_n > ...$ and $S(\mathbf{y}_{\mathbb{N}}) = \dfrac{S(\mathbf{x}_{\mathbb{N}})}{x_2^2} - \dfrac{1}{1+4x_2}$)
then $S(\mathbf{y}_{\mathbb{N}}) \geq S$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{1+4x_2} + x_2^2 S \implies$

$$S \geq \frac{1}{1+4x_2} + x_2^2 S \iff S \geq \frac{1}{(1+4x_2)(1-x_2^2)}.$$

We will find $\mu := \max\limits_{x\in(0,1)} h(x)$, where

$$h(x) := (1+4x)\left(1-x^2\right) = -4x^3 - x^2 + 4x + 1.$$

Since $h'(x) = -12x^2 - 2x + 4 = -2(3x+2)(2x-1)$ then
$\mu = \max\limits_{x\in(0,1)} h(x) = h\left(\dfrac{1}{2}\right) = \dfrac{9}{4}$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{\mu} = \dfrac{4}{9}$.
Since $S(\mathbf{x}_{\mathbb{N}}) = \dfrac{1}{(1+4q)(1-q^2)}$ for $x_n = q^{n-1}, n \in \mathbb{N}, q \in (0,1)$,
then for $q = \dfrac{1}{2}$ we obtain

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{\left(1 + 4\cdot\frac{1}{2}\right)\left(1 - \left(\frac{1}{2}\right)^2\right)} = \frac{4}{9}.$$

★ Problem 8.18(SSMJ 5345)
Indeed, $|a\cos x + b\cos y| \leq \sqrt{a^2 + b^2 + 2ab\cos(x+y)} \iff$
$(a\cos x + b\cos y)^2 \leq a^2 + b^2 + 2ab\cos(x+y) \iff$
$a^2\cos^2 x + b^2\cos^2 y + 2ab\cos x\cos y \leq a^2 + b^2 + 2ab\cos(x+y) \iff$
$0 \leq a^2\sin^2 x + b^2\sin^2 y + 2ab(\cos(x+y) - \cos x\cos y) \iff$
$0 \leq (a\sin x - b\sin y)^2$.
Equality occurs iff $a\sin x - b\sin y = 0 \iff a\sin x = b\sin y$.
Let $\varphi := x + y$,then $\sin x \div \sin y = \dfrac{1}{a} \div \dfrac{1}{b} = bc \div ca \iff$
$\sin x = kbc, \sin y = kca$ and $\sin y = \sin(\varphi - x) \iff$
$\sin y = \sin\varphi\cos x - \cos\varphi\sin x \iff kca = \sin\varphi\sqrt{1 - k^2b^2c^2} - kbc\cos\varphi \iff$

$(kca + kbc\cos\varphi)^2 = \sin^2\varphi - k^2b^2c^2\sin^2\varphi \iff$
$k^2c^2a^2 + k^2b^2c^2\cos^2\varphi + 2abc^2\cos\varphi = \sin^2\varphi - k^2b^2c^2\sin^2\varphi \iff$
$k^2c^2\left(a^2 + b^2 + 2ab\cos\varphi\right) = \sin^2\varphi \iff k^2 = \dfrac{\sin^2\varphi}{c^2\left(a^2 + b^2 + 2ab\cos\varphi\right)}$.
Hence, $\cos^2 x = 1 - k^2b^2c^2 = 1 - \dfrac{b^2\sin^2\varphi}{a^2 + b^2 + 2ab\cos\varphi} = \dfrac{(a + b\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$

and $\cos^2 y = 1 - k^2c^2a^2 = \dfrac{(b + a\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$.

(Obviously that $\dfrac{(a + b\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi} \le 1$ and $\dfrac{(b + a\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$).

★ Problem 8.19.

Solution 1.

Lemma 1.

For any positive real x and any natural n holds inequality

(1) $\quad \dfrac{x^n + x^{n-1} + ... + x}{n} \le \dfrac{x^{n+1} + 1}{2}$.

Proof.

Since $\left(x^{n+1-k} - 1\right)\left(x^k - 1\right) \ge 0, k = 1, 2, ..., n$ and $\sum\limits_{k=1}^{n}\left(x^{n+1-k} - 1\right)\left(x^k - 1\right) =$

$\sum\limits_{k=1}^{n}\left(x^{n+1} + 1 - x^k - x^{n+1-k}\right) = \sum\limits_{k=1}^{n}\left(x^{n+1} + 1\right) - \sum\limits_{k=1}^{n}\left(x^k + x^{n+1-k}\right) =$

$n\left(x^{n+1} + 1\right) - 2\sum\limits_{k=1}^{n} x^k$ then $n\left(x^{n+1} + 1\right) - 2\sum\limits_{k=1}^{n} x^k \ge 0 \iff (1)$.

Lemma 2.

For any positive real x and any natural n holds inequality

(2) $\quad \dfrac{x^n + x^{n-1} + ... + x + 1}{n + 1} \ge \left(\dfrac{x + 1}{2}\right)^n$.

Proof.(Math Induction by n**).**

For $n = 1$ inequality **(1)** obviously holds.

Let $n \in \mathbb{N}$. From supposition that **(2)** right follows

$\left(\dfrac{x + 1}{2}\right)^{n+1} \le \dfrac{x^n + x^{n-1} + ... + x + 1}{n + 1} \cdot \dfrac{x + 1}{2}$ and inequality **(1)** yields

$\dfrac{x^n + x^{n-1} + ... + x + 1}{n + 1} \cdot \dfrac{x + 1}{2} \le \dfrac{x^{n+1} + x^n + ... + x + 1}{n + 2}$.

Indeed,

$(n + 2)(x + 1)\left(x^n + x^{n-1} + ... + x + 1\right) \le 2(n + 1)\left(x^{n+1} + x^n + ... + x + 1\right) \iff$

$(n + 2)\left(x^{n+1} + 2x^n + ... + 2x + 1\right) \le 2(n + 1)\left(x^{n+1} + x^n + ... + x + 1\right) \iff$

$2(x^n + ... + x) \le n\left(x^{n+1} + 1\right)$.

Since by AM-GM Inequality $\sqrt[n]{n!} \le \dfrac{1 + 2 + ... + n}{n} = \dfrac{n + 1}{2} \iff$

$n! \le \left(\dfrac{n + 1}{2}\right)^n$ and by **(2)** $\dfrac{n^m + n^{m-1} + ... + n + 1}{m + 1} \ge \left(\dfrac{n + 1}{2}\right)^m$ then

$\left(\dfrac{n^m + n^{m-1} + ... + n + 1}{m + 1}\right)^n \ge \left(\dfrac{n + 1}{2}\right)^{mn} \ge (n!)^m$.

Solution 2.

Lemma 1.

For any $a \in [0, 1]$ and $n \in \mathbb{N}$ holds inequality

(3) $\quad (1 + a)^n - (1 - a)^n \ge 2na$.

with equality condition in both inequalities $a=0$ or $n=1$.

Proof.

By inequality Bernoulli $(1\pm a)^k \geq 1\pm ka$ for any $k=1,2,...n-1$ we obtain

$(1+a)^n-1=a\left((1+a)^{n-1}+(1+a)^{n-2}+...+(1+a)+1\right)\geq$

$a\left((1+(n-1)a)+(1+(n-2)a)+...+(1+a)+1\right)=a\left(n+a\left(1+2+...+(n-1)\right)\right)=$

$a\left(n+\dfrac{n(n-1)}{2}a\right)=an+\dfrac{n(n-1)}{2}a^2$ and

$1-(1-a)^n=a\left((1-a)^{n-1}\right)+(1-a)^{n-2}+...+(1-a)+1)\geq$

$a\left((1-(n-1)a)+(1-(n-2)a)+...+(1-a)+1\right)=a\left(n-a\left(1+2+...+(n-1)\right)\right)=$

$an-\dfrac{n(n-1)}{2}a^2.$

Thus, we have two inequalities

(4) $\quad (1+a)^n\geq 1+na+\dfrac{n(n-1)}{2}a^2$ and

(5) $\quad (1-a)^n\leq 1-na+\dfrac{n(n-1)}{2}a^2,$ with $a=0$ or $n=1$ as equality condition

in both inequalitiws.

For any natural n and any $a\in[0,1]$ from inequalities **(4)** and **(5)** immediately

follows inequality **(3)**.

Lemma 2.

For any natural n and any non-negative x and y holds inequality

(6) $\quad \dfrac{x^n+x^{n-1}y+...+xy^{n-1}+y^n}{n+1}\geq\left(\dfrac{x+y}{2}\right)^n.$

Proof.

Due symmetry we can suppose that $x\leq y$ and excluding trivial cases

$x=0$ and $x=y$ we assume that $0<x<y$.

Then for $a:=\dfrac{y-x}{y+x}$ holds $0<a<1$.

Plugging this a in inequality **(3)** we obtain

$\left(1+\dfrac{y-x}{y+x}\right)^{n+1}-\left(1-\dfrac{y-x}{y+x}\right)^{n+1}\geq 2(n+1)\dfrac{y-x}{y+x}\iff$

$\dfrac{2^{n+1}}{(x+y)^n}(y^n-x^n)\geq 2(n+1)\dfrac{y-x}{y+x}\iff\dfrac{y^n-x^n}{(n+1)(y-x)}\geq\left(\dfrac{x+y}{2}\right)^n\iff$

$\dfrac{x^n+x^{n-1}y+...+xy^{n-1}+y^n}{n+1}\geq\left(\dfrac{x+y}{2}\right)^n.$

In particularly, if $y=1$ we obtain inequality

(7) $\quad \dfrac{x^n+x^{n-1}+...+x+1}{n+1}\geq\left(\dfrac{x+1}{2}\right)^n.$

Since by **AM-GM** Inequality

$\sqrt[n]{n!}\leq\dfrac{1+2+...+n}{n}=\dfrac{n+1}{2}\iff n!\leq\left(\dfrac{n+1}{2}\right)^n$

and by **(7)** $\dfrac{n^m+n^{m-1}+...+n+1}{m+1} \geq \left(\dfrac{n+1}{2}\right)^m$ then
$\left(\dfrac{n^m+n^{m-1}+...+n+1}{m+1}\right)^n \geq \left(\dfrac{n+1}{2}\right)^{mn} \geq (n!)^m$.

Problem 8.20(Met. Rec).
Note that $(n+1)\cos\dfrac{\pi}{n+1} - n\cos\dfrac{\pi}{n} > 1 \iff$
$(n+1)\cos\dfrac{\pi}{n+1} - (n+1) > n\cos\dfrac{\pi}{n} - n \iff$
$\dfrac{n+1}{\pi}\cos\dfrac{\pi}{n+1} - \dfrac{n+1}{\pi} > \dfrac{n}{\pi}\cos\dfrac{\pi}{n} - \dfrac{n}{\pi}$.
let $g(x) := \dfrac{\cos x}{x} - \dfrac{1}{x}$.We will prove that $g(x)$ decrease on $(0,\pi/2)$.
Let $0 < x < x+h < \pi/2$.
Then $g(x+h) - g(x) = \dfrac{\cos(x+h)}{x+h} - \dfrac{1}{x+h} - \dfrac{\cos x}{x} + \dfrac{1}{x} =$
$\dfrac{x\cos(x+h) - (x+h)\cos x + h}{x(x+h)} = \dfrac{x(\cos(x+h) - \cos x) + h(1-\cos x)}{x(x+h)} =$
$\dfrac{-2x\sin\left(x+\dfrac{h}{2}\right)\sin\dfrac{h}{2} + 2h\sin^2\dfrac{x}{2}}{x(x+h)} < \dfrac{-2x\sin x\sin\dfrac{h}{2} + 2h\sin^2\dfrac{x}{2}}{x(x+h)} =$
$-\dfrac{4\sin x\left(\dfrac{h}{2}\tan\dfrac{x}{2} - \dfrac{x}{2}\sin\dfrac{h}{2}\right)}{x(x+h)} < 0$ because
$\dfrac{h}{2} > \sin\dfrac{h}{2}$ and $\tan\dfrac{x}{2} > \dfrac{x}{2}$.
Since $\dfrac{\pi}{n+1} < \dfrac{\pi}{n}$ then $g\left(\dfrac{\pi}{n+1}\right) > g\left(\dfrac{\pi}{n}\right) \iff$
$\dfrac{n+1}{\pi}\cos\dfrac{\pi}{n+1} - \dfrac{n+1}{\pi} > \dfrac{n}{\pi}\cos\dfrac{\pi}{n} - \dfrac{n}{\pi}$.

Finding maximum,minimum and range.

Problem 8.21(82-Met. Rec).
Solution 1.
First note that $\dfrac{x^2-2x+1}{6x^2-7x+3} = \dfrac{(x-1)^2}{6x^2-7x+3} \geq 0$ for all real x because discriminant of quadratic trinomial $6x^2-7x+3$ is negative.
Also note that $\min\limits_{x\in\mathbb{R}} \dfrac{-x^2+2x-1}{6x^2-7x+3} = -\max\limits_{x\in\mathbb{R}} \dfrac{(x-1)^2}{6x^2-7x+3} =$
$-\max\limits_{x\in\mathbb{R}\setminus\{1\}} \dfrac{(x-1)^2}{6x^2-7x+3}$ (because $\dfrac{(x-1)^2}{6x^2-7x+3} = 0 \iff x=1$ and
$\dfrac{(x-1)^2}{6x^2-7x+3} > 0$ for any $x \in \mathbb{R}\setminus\{1\}$).

Since $6x^2-7x+3=6(x-1)^2+5(x-1)+2$ then for $x\in\mathbb{R}\setminus\{1\}$, denoting $t:=\dfrac{1}{x-1}$, we obtain
$\dfrac{(x-1)^2}{6x^2-7x+3}=\dfrac{(x-1)^2}{6(x-1)^2+5(x-1)+2}=\dfrac{1}{6+5t+2t^2}$
and, therefore,
$\min\limits_{x\neq 1}\dfrac{-x^2+2x-1}{6x^2-7x+3}=-\max\limits_{t\neq 0}\dfrac{1}{6+5t+2t^2}=-\dfrac{1}{\min\limits_{t\neq 0}(6+5t+2t^2)}$.
Since $6+5t+2t^2=2(t+5/4)^2+23/8$ then $\min\limits_{t\neq 0}\left(6-5t+2t^2\right)=23/8$.
Thus, $\min\limits_{x\neq 1}\dfrac{-x^2+2x-1}{6x^2-7x+3}=-\dfrac{8}{23}$ and is reached if $\dfrac{1}{x-1}=-\dfrac{5}{4}\iff x=\dfrac{1}{5}$.

Solution 2.
Let $h(x):=\dfrac{x^2-2x+1}{6x^2-7x+3}$.
Since $Range\,(h(x))=\{t\mid t\in\mathbb{R}$ and $h(x)=t$ is solvable in $x\in\mathbb{R}\}$
and $6x^2-7x+3>0$ for any $x\in\mathbb{R}$ then
$h(x)=t\iff x^2-2x+1=t\left(6x^2-7x+3\right)\iff$
(1) $(6t-1)x^2-(7t-2)x+3t-1=0$.
If $t=1/6$ then equation **(1)** becomes $-\dfrac{5}{6}x-\dfrac{1}{2}=0\iff x=-\dfrac{3}{5}$.
So, $1/6\in Range\,(h(x))$.
If $t\neq 1/6$ then quadratic equation **(1)** solvable iff it's discrimianant
$D\geq 0$,that is iff $(7t-2)^2-4(6t-1)(3t-1)\geq 0\iff 8t-23t^2\geq 0\iff$
$t(23t-8)\leq 0\iff t\in[0,8/23]\setminus\{1/6\}$.
Thus, $Range\,(h(x))=[0,8/23]$ and, therefore,
$\min\limits_{x\in\mathbb{R}}\dfrac{-x^2+2x-1}{6x^2-7x+3}=--\max\limits_{x\in\mathbb{R}}h(x)=-\dfrac{8}{23}$.
Problem 8.22 (83-Met. Rec).
Note that

$$\max_{x,y>0}\{\min\{x,1/y,y+1/x\}\}=\max\{t\mid t>0 \text{ and } \exists(x,y>0)\,[x,1/y,y+1/x]\geq t\}.$$

Also

$$\min\{x,1/y,y+1/x\}\geq t\iff\begin{cases}x\geq t\\1/y\geq t\\y+1/x\geq t\end{cases}\iff\begin{cases}x\geq t\\y\leq 1/t\\t-1/x\leq y\\t-1/x\leq 1/t\end{cases}\iff$$

$$\begin{cases}t-1/t\leq 1/x\leq 1/t\\t-1/x\leq y\leq 1/t\end{cases}\implies t-1/t\leq 1/t\iff t^2\leq 2\iff t\leq\sqrt{2}.$$

$\max t=\sqrt{2}$ and attained if $x=\sqrt{2},y=1/\sqrt{2}$.

Problem 8.23(58-Met. Rec.).

Let $rem_k(n)$ be remainder from division n by k. We will prove that
$\max\left\{rem_k(n) \vdots k \in \{1,2,...,n\}\right\} = \left[\dfrac{n-1}{2}\right]$.

Note that $rem_k(n) = n - k\left[\dfrac{n}{k}\right]$ and $n = \left[\dfrac{n}{2}\right] + \left[\dfrac{n-1}{2}\right] + 1$.

Also note that suffices to prove inequality $rem_k(n) \le \left[\dfrac{n-1}{2}\right]$

for any $k \le \left[\dfrac{n}{2}\right]$.

Indeed, since $k \ge \left[\dfrac{n}{2}\right]+1 \implies k > \dfrac{n}{2} \iff n-k < k$ and $n = 1\cdot k+(n-k)$,

we have $rem_k(n) = n - k \le n - \left(\left[\dfrac{n}{2}\right]+1\right) = \left[\dfrac{n-1}{2}\right]$.

Let $1 \le k \le \left[\dfrac{n}{2}\right]$ then $rem_k(n) \le \left[\dfrac{n-1}{2}\right] \iff$

$$\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 1 - k\left[\frac{n}{k}\right] \le \left[\frac{n-1}{2}\right] \iff$$

$$\left[\frac{n}{2}\right] + 1 \le k\left[\frac{n}{k}\right] \iff \frac{\left[\frac{n}{2}\right]+1}{k} \le \left[\frac{n}{k}\right] \iff$$

$$\left[\frac{\left[\frac{n}{2}\right]+1+(k-1)}{k}\right] \le \left[\frac{n}{k}\right] \iff$$

$$\left[\frac{\left[\frac{n+2k}{2}\right]}{k}\right] \le \left[\frac{n}{k}\right] \iff \left[\frac{n+2k}{2k}\right] \le \left[\frac{n}{k}\right].$$

If $k=1$ latter inequality becomes $\left[\dfrac{n+2}{2}\right] \le n \iff \left[\dfrac{n}{2}\right]+1 \le n \iff$

$1 \le \left[\dfrac{n+1}{2}\right]$ and obviously holds for any $n \ge 2$.

If $2 \le k \le \left[\dfrac{n}{2}\right]$ then $\dfrac{n+2k}{2k} \le \dfrac{n}{k} \iff \dfrac{n+2k}{2} \le n \iff n+2k \le 2n \iff$
$k \le \left[\dfrac{n}{2}\right]$.

Thus, for any $k \in \{1,2,...,n\}$ holds inequality $rem_k(n) \le \left[\dfrac{n-1}{2}\right]$ and

since for $k = \left[\dfrac{n}{2}\right]+1$ we have $n = 1\cdot k + \left[\dfrac{n-1}{2}\right]$ and $\left[\dfrac{n-1}{2}\right] < k$ then

$\max rem_k(n) = \left[\dfrac{n-1}{2}\right]$.

Remark.
Here was used the following properties of $[x]$ (integer part of x):
1. $a \le b \implies [a] \le [b]$.
Proof.

Since $[a] \le a$ and $a \le b$ then $[a] \le b \implies [a] \in \left\{t \vdots t \in \mathbb{Z} \text{ and } t \le b\right\}$ and,

therefore, $[a] \le \max\left\{t \vdots t \in \mathbb{Z} \text{ and } t \le b\right\} = [b]$.

2. For any $n \in \mathbb{Z}$ holds identity $n = \left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right]$.

Proof.

If $n = 2k$ then $\left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right] = \left[\frac{2k}{2}\right] + \left[\frac{2k+1}{2}\right] =$

$k + \left(k + \left[\frac{1}{2}\right]\right) = 2k$;

If $n = 2k+1$ then $\left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right] = \left[\frac{2k+1}{2}\right] + \left[\frac{2k+2}{2}\right] =$

$\left(k + \left[\frac{1}{2}\right]\right) + k + 1 = 2k+1$.

3. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ holds identity $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$.

Proof.

$p := \left[\frac{x}{n}\right]$ then $p \le \frac{x}{n} < p+1 \iff np \le x < np+n$.

Since $np \in \left\{t \vdots t \in \mathbb{Z} \text{ and } t \le x\right\}$ then $np \le [x] = \max\left\{t \vdots t \in \mathbb{Z} \text{ and } t \le x\right\}$

and we have $np \le [x] \le x < np+n \implies$

$np \le [x] < np+n \iff p \le \frac{[x]}{n} < p+1 \iff \left[\frac{[x]}{n}\right] = p$.

Problem 8.24

Note that $F(x,y,z) = \max\{|\cos x| + |\cos 2y|,\ |\cos y| + |\cos 2z|,\ |\cos z| + |\cos 2x|\} \ge$

$$\frac{(|\cos x| + |\cos 2y|) + (|\cos y| + |\cos 2z|) + (|\cos z| + |\cos 2x|)}{3} =$$

$$\frac{(|\cos x| + |\cos 2x|) + (|\cos y| + |\cos 2y|) + (|\cos z| + |\cos 2z|)}{3} \ge \frac{M+M+M}{3}$$

where $M = \min_t (|\cos t| + |\cos 2t|) = |\cos t_0| + |\cos 2t_0|$ for some t_0.

So, $F(x,y,z) \ge M$ for all real x,y,z and $\min_{x,y,z} F(x,y,z) = M$ because $F(t_0,t_0,t_0) = M$ and for solving our problem we need to find $\min_t (|\cos t| + |\cos 2t|)$.

Denote $u := |\cos t|$, then $u \in [0,1]$ and $|\cos t| + |\cos 2t| = u + \left|2u^2 - 1\right|$.

There are many ways to find $\min_{u\in[0,1]} \left(u + \left|2u^2-1\right|\right)$, but I prefer the following way:

$u + \left|2u^2-1\right| = u + \left|u - \frac{1}{\sqrt{2}}\right| \left|2u + \sqrt{2}\right| \ge u + \left|u - \frac{1}{\sqrt{2}}\right| \ge \frac{1}{\sqrt{2}}$, because $2u + \sqrt{2} \ge \sqrt{2} > 1$ and for arbitrary real a,b inequality $a + |a-b| \ge b$ holds.

Let's compare this way with a traditional way:

$\left|2u^2-1\right| = \left\{\begin{array}{ll} 2u^2-1 & \text{for } \dfrac{1}{\sqrt{2}} \le u \le 1 \\ 1-2u^2 & \text{for } 0 \le u \le \dfrac{1}{\sqrt{2}} \end{array}\right| .$

Therefore $\min\limits_{u\in[0,1]} \left(u+\left|2u^2-1\right|\right) = \min\left\{\min_1 \left(2u^2+u-1\right), \min{}_2 \left(1+u-2u^2\right)\right\}$

1. Let $\dfrac{1}{\sqrt{2}} \le u \le 1,$ then $2u^2+u-1$ increases on $\left[\dfrac{1}{\sqrt{2}},1\right]$ (because x-coordinate of parabola's vertex is less than 0 and, therefore,

$\min_1 \left(2u^2+u-1\right) = 2\cdot\left(\dfrac{1}{\sqrt{2}}\right)^2 + \dfrac{1}{\sqrt{2}} - 1 = \dfrac{1}{\sqrt{2}};$

2. Let $0 \le u \le \dfrac{1}{\sqrt{2}}$, then $\min{}_2 \left(1+u-2u^2\right) = \min\left\{1, \dfrac{1}{\sqrt{2}}\right\} = \dfrac{1}{\sqrt{2}},$ because function $1+u-2u^2$ have only local maximum $\dfrac{1}{4}$ on the segment $\left[0, \dfrac{1}{\sqrt{2}}\right]$.

So minimum can be obtained on the boundaries of the segment.

.

Problem 8.25 (M1067 Kvant)

Solution 1.

First we will find numbers p, q such that $\dfrac{x}{1-x^2} \ge px+q \iff$

$x \ge (px+q)\left(1-x^2\right)$ for any positive x with equality for $x = \dfrac{1}{\sqrt{3}}.$

Let $h(x) := x-(px+q)\left(1-x^2\right)$. We claim $h\left(\dfrac{1}{\sqrt{3}}\right) = h'\left(\dfrac{1}{\sqrt{3}}\right) = 0.$

Then we have $h\left(\dfrac{1}{\sqrt{3}}\right) = 0 \iff \dfrac{1}{\sqrt{3}} = \left(\dfrac{p}{\sqrt{3}}+q\right)\left(1-\dfrac{1}{3}\right) \iff$

$\dfrac{1}{\sqrt{3}} = \left(\dfrac{p}{\sqrt{3}}+q\right)\dfrac{2}{3} \iff \dfrac{\sqrt{3}}{2} = \dfrac{p}{\sqrt{3}}+q;$

$h'\left(\dfrac{1}{\sqrt{3}}\right) = 0 \iff 1 = p\left(1-\dfrac{1}{3}\right) - 2\cdot\dfrac{1}{\sqrt{3}}\left(\dfrac{p}{\sqrt{3}}+q\right) \iff$

$1 = \dfrac{2p}{3} - 2\cdot\dfrac{1}{\sqrt{3}}\dfrac{\sqrt{3}}{2} \iff 2 = \dfrac{2p}{3} \iff p = 3 \implies \dfrac{\sqrt{3}}{2} = \dfrac{3}{\sqrt{3}}+q \iff q = -\dfrac{\sqrt{3}}{2}.$

And then $x-\left(3x-\dfrac{\sqrt{3}}{2}\right)\left(1-x^2\right) = 3\left(x+\dfrac{1}{2}\sqrt{3}\right)\left(x-\dfrac{1}{\sqrt{3}}\right)^2.$

So, $\dfrac{x}{1-x^2} \ge 3x-\dfrac{\sqrt{3}}{2}$ for any $x \ge 0$ and, therefore,

$\sum\limits_{cyc} \dfrac{x}{1-x^2} \ge \sum\limits_{cyc} \left(3x-\dfrac{\sqrt{3}}{2}\right) = 3(x+y+z) - \dfrac{3\sqrt{3}}{2}.$

Since $(x+y+z)^2 \ge 3(xy+yz+zx) = 3 \iff x+y+z \ge \sqrt{3}$

then $3(x+y+z) - \dfrac{3\sqrt{3}}{2} \ge 3\sqrt{3} - \dfrac{3\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2}.$

Solution 2.

Since $x,y,z\in(0,1)$ then $\sum_{cyc}\dfrac{x}{1-x^2}=\sum_{cyc}\sum_{k=1}^{\infty}x^{2k-1}=\sum_{k=1}^{\infty}\sum_{cyc}x^{2k-1}$
and by PM-AM Inequality
we have $\dfrac{1}{3}\sum_{cyc}x^{2k-1}\geq\left(\dfrac{x+y+z}{3}\right)^{2k-1}\iff\sum_{cyc}x^{2k-1}\geq\dfrac{(x+y+z)^{2k-1}}{3^{2(k-1)}}$.
Also we have $(x+y+z)^2\geq 3(xy+yz+zx)=3\iff x+y+z\geq\sqrt{3}$.
Then $\sum_{cyc}x^{2k-1}\geq\dfrac{\left(\sqrt{3}\right)^{2k-1}}{3^{2(k-1)}}=\dfrac{3\sqrt{3}}{3^k}$ and, therefore,
$\sum_{cyc}\dfrac{x}{1-x^2}\geq\sum_{k=1}^{\infty}\dfrac{3\sqrt{3}}{3^k}=3\sqrt{3}\cdot\dfrac{1/3}{1-1/3}=\dfrac{3\sqrt{3}}{2}$.

Solution 3.
Let $h(x)=\dfrac{x}{1-x^2}$.Since $h'(x)=\dfrac{1}{(1-x)^2}+\dfrac{1}{(1+x)^2}>0$ and
$h''(x)=\dfrac{2}{(1-x)^3}-\dfrac{2}{(1+x)^3}=\dfrac{4x\cdot\left(x^2+3\right)}{(1-x)^3(x+1)^3}>0$ for $x\in(0,1)$
then $h(x)$ is concave up and increasing on $(0,1)$.
Hence, by Jensen's Inequality
$$\frac{h(x)+h(y)+h(z)}{3}\geq h\left(\frac{x+y+z}{3}\right).$$
Since $(x+y+z)^2\geq 3(xy+yz+zx)=3\iff x+y+z\geq\sqrt{3}$ and
$h(x)$ increasing on $(0,1)$ then
$h\left(\dfrac{x+y+z}{3}\right)\geq h\left(\dfrac{\sqrt{3}}{3}\right)=\dfrac{\sqrt{3}}{2}$ and, therefore, $h(x)+h(y)+h(z)\geq\dfrac{3\sqrt{3}}{2}$.

Solution 4.
By Cauchy Inequality $\sum_{cyc}\dfrac{x}{1-x^2}=\sum_{cyc}\dfrac{x^2}{x-x^3}\geq\dfrac{(x+y+z)^2}{x+y+z-(x^3+y^3+z^3)}=$
.
Since $(x+y+z)^2\geq 3(xy+yz+zx)=3\iff x+y+z\geq\sqrt{3}$ and
$x^3+y^3+z^3\geq\dfrac{(x+y+z)^3}{9}\geq\dfrac{x+y+z}{3}$ then $\dfrac{x^3+y^3+z^3}{x+y+z}\geq\dfrac{1}{3}$
and $\dfrac{(x+y+z)^2}{x+y+z-(x^3+y^3+z^3)}=\dfrac{x+y+z}{1-\dfrac{x^3+y^3+z^3}{x+y+z}}\geq\dfrac{\sqrt{3}}{1-\dfrac{1}{3}}=\dfrac{3\sqrt{3}}{2}$.

★Problem 8.26**
Let $n=3$. We have $\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}+\dfrac{1}{1+x_3}=1\iff$
$3+2(x_1+x_2+x_3)+x_1x_2+x_2x_3+x_3x_1=1+x_1+x_2+x_3+x_1x_2+x_2x_3+$
$x_3x_1+x_1x_2x_3\iff 2+x_1+x_2+x_3=x_1x_2x_3$.
Since $x_1+x_2+x_3\geq 3\sqrt[3]{x_1x_2x_3}$ then $x_1x_2x_3\geq 2+3\sqrt[3]{x_1x_2x_3}\iff$
$\left(\sqrt[3]{x_1x_2x_3}-2\right)\left(\sqrt[3]{x_1x_2x_3}+1\right)^2\geq 0\iff$
$\sqrt[3]{x_1x_2x_3}-2\geq 0\iff x_1x_2x_3\geq 2^3$.
Or, by another way:
Since $\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}+\dfrac{1}{1+x_3}=1\iff\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}=\dfrac{x_3}{1+x_3}\iff$

$\dfrac{1+x_3}{1+x_1}+\dfrac{1+x_3}{1+x_2}=x_3 \implies x_3 \geq 2(1+x_3)\sqrt{\dfrac{1}{1+x_1}.\dfrac{1}{1+x_2}}=\dfrac{2(1+x_3)}{\sqrt{(1+x_1)(1+x_2)}}.$

Similarly we obtain $x_2 \geq \dfrac{2(1+x_2)}{\sqrt{(1+x_3)(1+x_1)}}, x_1 \geq \dfrac{2(1+x_1)}{\sqrt{(1+x_2)(1+x_3)}}.$

Hence,

$x_1x_2x_3 \geq \dfrac{2^3(1+x_1)(1+x_2)(1+x_3)}{\sqrt{(1+x_2)(1+x_3)}\cdot\sqrt{(1+x_3)(1+x_1)}\cdot\sqrt{(1+x_1)(1+x_2)}}=2^3.$

Using idea of this way we can prove general case.

We have for any $k=1,2,...,n$

$\sum_{i=1}^{n}\dfrac{1}{1+x_i}=1 \iff \sum_{i=1,i\neq k}^{n}\dfrac{1}{1+x_i}=\dfrac{x_k}{1+x_k} \iff \sum_{i=1,i\neq k}^{n}\dfrac{1+x_k}{1+x_i}=x_k.$

Then by AM-GM Inequality

$x_k=\sum_{i=1,i\neq k}^{n}\dfrac{1+x_k}{1+x_i}\geq (n-1)\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}\dfrac{1+x_k}{1+x_i}} \iff$

$a_k \geq \dfrac{(n-1)(1+x_k)}{\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}(1+x_i)}},\ k=1,2,...,n.$

Let $P:=\prod_{k=1}^{n}(1+x_k)$. Since $\prod_{i=1,i\neq k}^{n}(1+x_i)=\dfrac{P}{1+x_k}$ then

$\prod_{k=1}^{n}\prod_{i=1,i\neq k}^{n}(1+x_i)=\dfrac{P^n}{\prod_{k=1}^{n}(1+x_k)}=P^{n-1}$ and, therefore,

$$\prod_{k=1}^{n}x_k \geq \prod_{k=1}^{n}\frac{(n-1)(1+x_k)}{\sqrt[n-1]{\prod_{i=1,i\neq k}^{n}(1+x_i)}}=\frac{(n-1)^n\prod_{k=1}^{n}(1+x_k)}{\sqrt[n-1]{\prod_{k=1}^{n}\prod_{i=1,i\neq k}^{n}(1+x_i)}}=\frac{(n-1)^nP}{\sqrt[n-1]{P^{n-1}}}=(n-1)^n$$

.

9. Invariants.

Problem 9.1(65-Met. Rec.).

a) If in initial fraction $\dfrac{a}{b}$ parity of numerator and denominator is different then parity of numerator and denominator of the fraction after transformations

$\dfrac{a}{b}\mapsto\dfrac{a-b}{b},\dfrac{a+b}{b},\dfrac{b}{a}$ remains different as well.

Hence, starting with fraction 1/2 and using such transformation the fraction 67/91 can′t be obtained.

b) By the same reason as in **a)** can't be obtained the pair $(5/6, 9/11)$.

c) Note that number $S(a,b,c)=a^2+b^2+c^2$ is invariant of allowed transformations. Indeed, let (a,b,c) be transformed to $\left(\dfrac{a+b}{\sqrt{2}},\dfrac{a-b}{\sqrt{2}},c\right)$ then

$$S\left(\frac{a+b}{\sqrt{2}},\frac{a-b}{\sqrt{2}},c\right)=\left(\frac{a+b}{\sqrt{2}}\right)^2+\left(\frac{a-b}{\sqrt{2}}\right)^2+c^2=a^2+b^2+c^2=S(a,b,c).$$

Since $S\left(\left(2,\sqrt{2},1/\sqrt{2}\right)\right)=4+2+\frac{1}{2}=\frac{13}{2}$ and
$S\left(1,\sqrt{2},\sqrt{2}-1\right)=1+2+3-2\sqrt{2}=6-2\sqrt{2}$
then to obtain the triple $\left(1,\sqrt{2},\sqrt{2}-1\right)$ isn't possible..

Problem 9.2(66-Met. Rec.).

Let a triple (a,b,c) of non-negative integer numbers represent state of population of chameleons on Rainbow Island, namely a,b, and $c-$are numbers of red, green and yellow chameleons, respectively. Then $(a_0,b_0,c_0)=(13,15,17)$ is initial population of chameleons, (a,b,c) is a current population.
Let (a_f,b_f,c_f) be prospective final population of chameleons, that is $(a_f,b_f,c_f)=(15,15,15)$.
Meeting of 2 chameleons of different colors we will call *productive* meeting.
Let $T_i(a,b,c), i=1,2,3$ be population after *productive* meeting of chameleons from population (a,b,c).
Possible three kind of transformation of (a,b,c) : .
$T_1(a,b,c)=(a-1,b-1,c+2)$ or $T_2(a,b,c)=(a-1,b+2,c-1)$ or
$T_3(a,b,c)=(a+2,b-1,c-1)$.
We will consider states of chameleon's populations by modulo 3.
Note that $(a_0,b_0,c_0)\equiv(1,0,-1)\pmod 3$ and
$T_i(a,b,c)\equiv(a-1,b-1,c-1)\pmod 3, i=1,2,3$.
Let $(a_i,b_i,c_i)=T_i(a,b,c), i=1,2,3$.
Since $a_i+b_i+c_i\equiv a+b+c\pmod 3$ then $rem_3(a+b+c)$ is invariant of transformations $T_i, i=1,2,3$.But this invariant isn't sensitive enough because $rem_3(a_0+b_0+c_0)=0$ and
$rem_3(a_f+b_f+c_f)=0$ as well (coincidence of initial and final states doesn't mean that initial state can be somehow transformed by $T_i, i=1,2,3$ to the final state).
Thus and so we will consider only states of chameleon's population which satisfy $a+b+c\equiv 0\pmod 3$.
Then $a_i^2+b_i^2+c_i^2=(a-1)^2+(b-1)^2+(c-1)^2=$
$a^2+b^2+c^2-2(a+b+c)+3\equiv a^2+b^2+c^2\pmod 3$
and this new invariant is sensitive because
$a_0^2+b_0^2+c_0^2\equiv\left(1^2+0^2+2^2\right)\pmod 3\equiv 2\pmod 3$ but
$a_f^2+b_f^2+c_f^2\equiv 0\pmod 3$.
Another simple invariant $R(a,b,c)=\{rem_3(x)\mid x\in\{a,b,c\}\}$ (set of remainders)
is better.
Indeed, since $R(a_i,b_i,c_i)=\{rem_3(x-1)\mid x\in\{a,b,c\}\}=R(a,b,c)$ and
$R(a_0,b_0,c_0)=\{0,1,2\}\neq R(a_0,b_0,c_0)=\{0\}$.

So, prospective final population of chameleons is impossible.
Analysis.
Invariants can help in the proof that some final state of the system can't be reached by admissible transformations from given initial state if value of invariant for both are different.
But in case of their coincidence the question remains open.
In some such problem possible use linear model for representation state of the system as, for example, in the recent problem, and then we can get answer not only for richness of the final state, but also how to get this state using admissible transformations.
Suppose that we apply k_i transformation of kind $T_i, i = 1, 2, 3$ to initial state (a_0, b_0, c_0) to obtain final state (a_f, b_f, c_f) ,that is
$(a_f, b_f, c_f) = (a_0, b_0, c_0) + k_1(-1, -1, 2) + k_2(-1, 2, -1) + k_3(2, -1, -1) \iff$

$$\begin{cases} -k_1 - k_2 + 2k_3 = a_f - a_0 \\ -k_1 + 2k_2 - k_3 = b_f - b_0 \\ 2k_1 - k_2 - k_3 = c_f - c_0 \end{cases} \iff \begin{cases} -k_1 - k_2 + 2k_3 = -2 \\ -k_1 + 2k_2 - k_3 = 0 \\ 2k_1 - k_2 - k_3 = 2 \end{cases}.$$

Easy to see that latter system have no solutions in integers.
Indeed, $2k_1 - k_2 - k_3 - (-k_1 + 2k_2 - k_3) = 2 \iff 3(k_1 - k_2) = 2$ and that impossible.
Let $p := a_f - a_0, q := b_f - b_0, r := c_f - c_0$.
For which, p, q, r the system has nonnegative integer solution.
First claim is obvious: $p + q + r = 0$.

Then $$\begin{cases} -k_1 - k_2 + 2k_3 = p \\ -k_1 + 2k_2 - k_3 = q \\ 2k_1 - k_2 - k_3 = r \end{cases} \iff \begin{cases} -k_1 - k_2 + 2k_3 = p \\ -k_1 + 2k_2 - k_3 = q \end{cases} \iff$$

$$\begin{cases} -k_2 + 2k_3 = p + k_1 \\ 2k_2 - k_3 = q + k_1 \end{cases} \iff \begin{cases} 3k_2 = p + 2q + 3k_1 \\ 3k_3 = 2p + q + 3k_1 \end{cases}.$$

So, claim number two: $p \equiv q \pmod 3$.
If $p \equiv q \pmod 3$ and $p + q + r = 0$ then for big enough non-negative integer k_1
we obtain nonnegative integer k_2 and k_3.
Apply represented above idea for solving the following training

Problem 9.3.
For solving this problem we will use represented above idea.
Suppose that we apply $k_i \in \mathbb{N} \cup \{0\}$ transformation of kind $T_i, i = 1, 2, 3, 4$ to initial state $(13, 17)$ to obtain final state $(37, 43)$.
Then correspondent Linear Model of this problem is:
$(37, 43) = (13, 17) + k_1(2, -1) + k_2(1, 2) + k_3(-2, 1) + k_4(-1, -2) \iff$

$$\begin{cases} 2k_1 + k_2 - 2k_3 - k_4 = 37 - 13 \\ -k_1 + 2k_2 + k_3 - 2k_4 = 43 - 17 \end{cases} \iff \begin{cases} 2k_1 + k_2 - 2k_3 - k_4 = 24 \\ -k_1 + 2k_2 + k_3 - 2k_4 = 26 \end{cases}.$$

Since $2k_1 + k_2 - 2k_3 - k_4 + 2(-k_1 + 2k_2 + k_3 - 2k_4) = 24 + 2 \cdot 26 \iff$
$5(k_2 - k_4) = 76$ then system have no integer solutions and, therefore, final state

of balls in the box isn't possible.

10. Miscellaneous problems.

Problem 10.1(1-Met. Rec.)
Let x_i be amount of mushrooms from the forest brought by $i-th$ pupil.
Since no two of them have not brought equally mushrooms we can assume that $x_1 > x_2 > ... > x_8$.
According to the condition of the problem $x_1 + x_2 + ... + x_8 = 60$. Thus,

(1) $\begin{cases} x_1 + x_2 + ... + x_8 = 60 \\ 1 \leq x_1 < x_2 < ... < x_8. \end{cases}$

and we will prove that $x_8 + x_7 + x_6 \geq x_1 + x_2 + ... + x_5 \iff$
$2(x_8 + x_7 + x_6) \geq x_1 + x_2 + ... + x_8 \iff$
$2(x_8 + x_7 + x_6) \geq 60 \iff x_8 + x_7 + x_6 \geq 30.$
Further we will consider four variants of proving this inequality.
Let $p := x_8 + x_7 + x_6, q := x_1 + x_2 + ... + x_5$ and $t := x_6$.
Since $p \geq t + t + 1 + t + 2 = 3t + 3$ and
$q \geq 1 + 2 + 3 + 4 + 5 = 15$ then $p = 60 - q \leq 60 - 15 = 45$. Also note that
$q \leq t - 1 + t - 2 + t - 3 + t - 4 + t - 5 = 5t - 15 \iff$
$60 - p \leq 5t - 15 \iff 75 - 5t \leq p.$

Hence, $\begin{cases} 3t + 3 \leq p \\ 75 - 5t \leq p \end{cases} \iff \max\{3t + 3, 75 - 5t\} \leq p$

and we have to prove that $p \geq 30$.
Using these notations we will consider following three variants of proving inequality $p \geq 30$.

Variant 1.
Since $3t + 3 \leq p \iff t \leq \dfrac{p-3}{3}$ and $75 - 5t \leq p \iff \dfrac{75-p}{5} \leq t$
then $\dfrac{75-p}{5} \leq t \leq \dfrac{p-3}{3}$ yields $\dfrac{75-p}{5} \leq \dfrac{p-3}{3} \iff$
$15 + 1 \leq \dfrac{p}{3} + \dfrac{p}{5} \iff 15 \cdot 16 \leq 8p \iff 30 \leq p.$

Remark.
Numbers $3, 8, 60$ in the problem so well matched, that p which provide solvability of inequality $\dfrac{75-p}{5} \leq t \leq \dfrac{p-3}{3}$ in $t \in \mathbb{R}$ automatically provide solvability of this inequality in $t \in \mathbb{N}$.
Rigorously, for integer t must be $\dfrac{75-p}{5} \leq t \leq \dfrac{p-3}{3} \iff$
$\left[\dfrac{75-p+4}{5}\right] \leq t \leq \left[\dfrac{p-3}{3}\right] \iff \left[\dfrac{79-p}{5}\right] \leq t \leq \left[\dfrac{p}{3}\right] - 1$ and for $t \geq 1$
we obtain $\max\left\{\left[\dfrac{79-p}{5}\right], 1\right\} \leq t \leq \left[\dfrac{p}{3}\right] - 1.$

Criteria solvability of latter inequality in $t \in \mathbb{N}$ is $\begin{cases} 2 \leq \left[\dfrac{p}{3}\right] \\ \left[\dfrac{79-p}{5}\right] \leq \left[\dfrac{p}{3}\right] - 1 \end{cases} \iff$

(2) $\begin{cases} 6 \le p \\ \left[\dfrac{79-p}{5}\right] \le \left[\dfrac{p}{3}\right] - 1 \end{cases}$.

Let $\left[\dfrac{79-p}{5}\right] = n \iff 5n \le 79 - p \le 5n + 4$ then $n + 1 \le \left[\dfrac{p}{3}\right] \iff 3n + 3 \le p$

and, therefore,

(2) $\iff \begin{cases} 6 \le p \\ 3n + 3 \le p \\ 5n \le 79 - p \le 5n + 4 \end{cases} \iff \begin{cases} 6 \le p \\ 3n + 3 \le p \\ 75 - 5n \le p \le 79 - 5n \end{cases} \iff$

$\begin{cases} \max\{6, 3n+3, 75-5n\} \le p \le 79 - 5n \\ 6 \le 79 - 5n \\ 3n + 3 \le 79 - 5n \end{cases} \iff \begin{cases} \max\{6, 3n+3, 75-5n\} \le p \le 79 - 5n \\ 5n \le 73 \\ 8n \le 76 \end{cases} \iff$

$\begin{cases} \max\{6, 3n+3, 75-5n\} \le p \le 79 - 5n \\ n \le 9 \end{cases} \iff \begin{cases} 75 - 5n \le p \le 79 - 5n \\ n \le 9 \end{cases}$

because

$75 - 5n \ge 6$ and $75 - 5n \ge 3n + 3$ for $n \le 9$.

Hence, $\min p = \min\limits_{n \le 9}(75 - 5n) = 30$.

Variant 2.

To solve problem suffice to prove that $\max\{3t+3, 75-5t\} \ge 30$,

for any natural t.

Since $\max\{3t+3, 75-5t\} \ge 30 \iff \left[\begin{array}{l} 3t + 3 \ge 30 \\ 75 - 5t \ge 30 \end{array}\right. \iff \left[\begin{array}{l} t \ge 9 \\ 9 \ge t \end{array}\right.$

then inequality $\max\{3t+3, 75-5t\} \ge 30$ holds for any natural t.

Variant 3.

Let $\varphi(t) := \max\{3t+3, 75-5t\}$ ..Since $3t + 3 \ge 75 - 5t \iff t \ge 9$ then

$\min\limits_{n \in \mathbb{N}} \varphi(t) = \begin{cases} 3t + 3 \text{ if } t \ge 9 \\ 75 - 5t \text{ if } t \le 9 \end{cases}$ and, therefore, $\min\limits_{t \in \mathbb{N}} \varphi(t) = \min\left\{\min\limits_{t \le 9} \varphi(t), \min\limits_{t \ge 9} \varphi(t)\right\} =$

$\min\left\{\min\limits_{t \le 9}(75 - 5t), \min\limits_{t \ge 9}(3t + 3)\right\} = \min\{30, 30\} = 30$.Hence, $30 \le p$.

Variant 4.

Let $t_1 := x_1$and $t_i := x_i - x_{i-1}, i = 2, 3, ..., 8$. Then

$x_1 = t_1, x_i = t_1 + t_2 + ... + t_i, i = 2, ..., 8$

where $t_1, t_2, ..., t_8 \in \mathbb{N}$ and $x_1 + x_2 + ... + x_8 = 60 \iff$

$8t_1 + 7t_2 + 6t_3 + 5t_4 + 4t_5 + 3t_6 + 2t_7 + t_8 = 60 \iff$

$4(x_1 + x_2) + 3t_2 + 6t_3 + 5t_4 + 4t_5 + 3t_6 + 2t_7 + t_8 = 60 \iff$

$x_1 + x_2 = \dfrac{60 - (3t_2 + 6t_3 + 5t_4 + 4t_5 + 3t_6 + 2t_7 + t_8)}{4} \implies$

$x_1 + x_2 \le \dfrac{60 - (3 \cdot 1 + 6 \cdot 1 + 5 \cdot 1 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 + 1)}{4} = 9$.

We have

$x_8 + x_7 + x_6 - (x_1 + x_2 + x_3 + x_4 + x_5) = t_8 + 2t_7 + 3(t_6 + t_5 + ... + t_1) -$

$(t_5 + 2t_4 + 3t_3 + 4t_2 + 5t_1) = t_8 + 2t_7 + 3t_6 + 2t_5 + t_4 - t_2 - 2t_1 =$

$(t_8 + 2t_7 + 3t_6 + 2t_5 + t_4) - (x_1 + x_2) \ge (1 + 2 + 3 + 2 + 1) - 9 = 0$

Variant 5. (Combinatorial solution).

Assume that there are no three pupils, whose collect amount of mushrooms not less than the other five pupils. That is for any $1 \leq i < j < k \leq 8$ holds inequality

$x_i + x_j + x_k < 30$. Summing all these inequalities we obtain

$(x_1 + x_2 + ... + x_8)\binom{8}{2} < 30\binom{8}{3} \iff 60 \cdot 28 < 30 \cdot 56 \iff 1 < 1$ that is contradiction.

Analysis of solutions.

Of course, the solution of the original problem is not exhausted by the above variants. There are very "childish" solutions in which long and unconvincing verbal periods are designed to replace the missing algebraic technique with branched logic.And the latter in such cases is not less (if not larger) stone of a stumbling block.But the choice for the given variants of the solution fell also because they represent some technique, the scope and utility of which is by no means confined to this problem.

Here the following ideas and techniques are involved:

1. $\max\limits_{x \in D} f(x) = \max\left\{\max\limits_{x \in D_1} f(x), \max\limits_{x \in D_2} f(x)\right\}$, where $D = D_1 \cup D_2$ (Similarly for $\min\limits_{x \in D} f(x)$).

2. Lover and upper bounds and attainable lover and upper bounds as minimum and maximum, respectively.

3. Reduction of extremal problems to parametrical (finding range of parameter which provides solvability of systems of inequalities).

4. Solving inequalities with integer parts.

5. Reduction a problem with dependent variables to the problem with independent variables.

Problem 10.2(2-Met. Rec.)

Let n is number of baskets and x_i is number of apples in i-th basket, $i = 1, 2, ..., n$ numbered so that $x_1 \geq x_2 \geq ... \geq x_n \geq 1$.

Suppose that the required situation is attainable that is remains k baskets and in k-th basket it is as many apples that if from i-th basket to throw δ_i apples, $i = 1, 2, ..., k$ then

$x_1 - \delta_1 = x_2 - \delta_2 = ... = x_k - \delta_k$ and

$x_1 - \delta_1 + x_2 - \delta_2 + ... + x_k - \delta_k = k(x_k - \delta_k) \geq 100 \implies kx_k \geq 100$.

Suppose now that there is k such that $kx_k \geq 100$.

Then for $\delta_i := x_i - x_k$, $i = 1, 2, ..., k-1$ and $\delta_k := 0$ we obtain

$x_i - \delta_i = x_k, i = 1, 2, ..., k$.

Therefore, $x_1 - \delta_1 + x_2 - \delta_2 + ... + x_k - \delta_k = kx_k \geq 100$.

Thus, existance of such k that $kx_k \geq 100$ is sufficient and necessity condition which provides claims of the problem

. We will prove that there is k for which $kx_k \geq 100$.

Assume contrary that $kx_k < 100$ for any $k = 1, 2, ..., n$.

Then in particular $1 \leq x_n < \dfrac{100}{n} \implies n < 100 \iff n \leq 99$ and

$2000 = x_1 + x_2 + ... + x_n < 100\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{99}\right) \implies$

$20 < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{99}$.

But $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{99} < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \left(\frac{1}{7} + \frac{1}{7} + ... + \frac{1}{7}\right) <$

$\frac{49}{20} + \frac{1}{7} \cdot 98 < 17$ and that is contradiction.

Analysis.

More precise estimation of the sum $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{99} < 6$ give the opportunity to solve the problem for common number of apples that not less then 600.

Problem 10.3(3-Met. Rec.)

Note that $3^0 \equiv 1 \pmod{10}, 3^1 \equiv 3 \pmod{10}, 3^2 \equiv 9 \pmod{10}, 3^3 \equiv 7 \pmod{10}$, $3^4 \equiv 1 \pmod{10}$ and so on...Let $n = 4k + r$,where $r = 0, 1, 2, 3, 4$.

Then for $r = 0, 1, 2, 3, 4$ the unite digit will be $1, 3, 9, 7$ respectively.

To prove that digit of tens in 3^n is even number suffices to prove that $3^n - r_{10}(3^n)$ divisible by 20. ($r_b(a)$ is remainder from division a by b).

We have $3^n - r_{10}(3^n) = \begin{cases} 3^{4k} - 1 \text{ (if } r = 0) \\ 3^{4k+1} - 3 \text{ (if } r = 1) \\ 3^{4k+2} - 9 \text{ (if } r = 2) \\ 3^{4k+3} - 7 \text{ (if } r = 3) \end{cases}$.

Note, that $3^{4k} - 1 = \left(3^4\right)^k - 1 \,\vdots\, 3^4 - 1 \implies 3^{4k} - 1 \,\vdots\, 20, \quad 3^{4k+1} - 3 = 3\left(3^{4k} - 1\right) \,\vdots\, 20$

and $3^{4k+2} - 9 = 9\left(3^{4k} - 1\right) \,\vdots\, 20$.

Since $3^{4k+3} - 7 \equiv 3^{4k+3} - 27 \pmod{20} \equiv 27\left(3^{4k} - 1\right) \pmod{20} \equiv 0 \pmod{20}$.

Problem 10.4(7-Met. Rec.)

Formulation of the problem eqivalent to the following:

Does it exist a natural number n such that $\left[10^8\left\{\sqrt{n}\right\}\right] = 19851986$?

Since $\left[10^8\left\{\sqrt{n}\right\}\right] = 19851986 \iff 19851986 < 10^8\left\{\sqrt{n}\right\} < 19851987 \iff$

$\frac{19851986}{10^8} < \left\{\sqrt{n}\right\} < \frac{19851987}{10^8}$($19851986 \neq 10^8\left\{\sqrt{n}\right\}$ because $\sqrt{n}$ either integer or irrational).

In the notation $\alpha := \frac{19851986}{10^8}, \beta := \frac{19851987}{10^8}$ latter inequality becomes $\alpha < \left\{\sqrt{n}\right\} < \beta$ and problem's question can be formulated more general:

Let $(\alpha, \beta) \subset (0, 1)$ be any interval. Does it exist a natural number n such that $\alpha < \left\{\sqrt{n}\right\} < \beta$?

It turns that unswer on this general question is positive and, in particular, positive for the original problem.

Indeed, denoting $p := [\sqrt{n}]$ we obtain

$\alpha < \{\sqrt{n}\} < \beta \iff \alpha < \sqrt{n} - p < \beta \iff (\alpha + p)^2 < n < (\beta + p)^2$.
And now the question arises:
When interval (a, b) contain an integer with guarantee?
The answer is quite simple:
If $b - a > 1$ then there is integer n that $a < n < b$.
Indeed, since $a < [a] + 1 \le a + 1 < b$ then $n = [a] + 1 \in (a, b)$.
(Of course condition $b - a > 1$ is only sufficient because
for example $0.9 < 1 < 1.01$).
Coming back to inequality $(\alpha + p)^2 < n < (\beta + p)^2$ we claim

$$(\beta + p)^2 - (\alpha + p)^2 > 1 \iff 2p(\beta - \alpha) > 1 - (\beta^2 - \alpha^2) \iff p > \frac{1 - (\beta^2 - \alpha^2)}{2(\beta - \alpha)}$$

.

Let p be any natural number satisfying latter inequality then interval $\left((\alpha + p)^2, (\beta + p)^2\right)$ contain at least one natural n, that is there are p, n natural such that $\alpha + p < \sqrt{n} < \beta + p \iff \alpha < \sqrt{n} - p < \beta$ and since $p < \alpha + p, \beta + p < p + 1$ then $p < \sqrt{n} < p + 1 \iff [\sqrt{n}] = p$.
Thus, $\alpha < \{\sqrt{n}\} < \beta$, Q.E.D.

Remark.

This problem can be solved by another way, but preference was given to represented solution because it clarify deep roots of original problem and allow solve the more general problem, introduce to wery useful technics and facts, leads to important concept of "dense set".
We say that proper subset $D \subset (0, 1)$ dence in $(0, 1)$ ﬀ for any $(\alpha, \beta) \subset (0, 1)$ there is $d \in D$ such that $d \in (\alpha, \beta)$,or by the other words if $D \cap (\alpha, \beta) \ne \varnothing$ for any $(\alpha, \beta) \subset (0, 1)$.
As a training exercise proposed the following

Problem.

Prove for any real numbers $\alpha < \beta$ there are $n, m \in \mathbb{N}$ such that
$\alpha < \sqrt[3]{n} - \sqrt{m} < \beta$.

Problem 10.5(12-Met. Rec.)

Solution 1.

Let $P(x) := ax^2 + bx + c$.Note that
$P(0) = c, P(2/3) = \dfrac{4a}{9} + \dfrac{2b}{3} + c = \dfrac{4a + 6b + 9c}{9}$
and $\dfrac{1}{3}P(0) + P(2/3) = \dfrac{c}{3} + \dfrac{4a + 6b + 9c}{9} = \dfrac{2(2a + 3b + 6c)}{9} = 0$.
So, $P(2/3) = -\dfrac{1}{3}P(0)$ .If $P(0)$ then $P(2/3) = 0$ as well and $2/3 \in (0, 1)$;
If $P(0) \ne 0$ then $P(0) \cdot P(2/3) < 0$ and, therefore, due continuity of $P(x)$ equation $P(x) = 0$ has solution in $(0, 2/3) \subset (0, 1)$.
Or, since $P(1) + 3P(1/3) = a + b + c + 3\left(\dfrac{a}{9} + \dfrac{b}{3} + c\right) = \dfrac{2(2a + 3b + 6c)}{3} = 0$
then $P(1/3) = 0$ or we have solution between $1/3$ and 1.

(But of course this does not mean that on $(0,1)$ we have two roots because $(0,2/3)\cap(1/3,1)\neq\varnothing$ and root can be the same).

Solution 2.
Let $F(x):=\dfrac{ax^3}{3}+\dfrac{bx^2}{2}+cx$ be primitive function for $P(x)$,that is
$F'(x)=P(x)$. Since $F(1)=\dfrac{a}{3}+\dfrac{b}{2}+c=\dfrac{2a+3b+6c}{6}=0=F(0)$
then by Roll's Theorem there is a point $x_0\in(0,1)$ such that
$F'(x_0)=0 \iff P(x_0)=0.$

Remark. Easy to prove that $2a+3b+6c=0$ imply existence of root of $P(x)$.
Indeed, since $b=-\dfrac{2a}{3}-2c$ then
$b^2-4ac=\left(-\dfrac{2a}{3}-2c\right)^2-4ac=\dfrac{4a^2-12ac+9c^2+27c^2}{9}=\dfrac{(2a-3c)^2}{9}+3c^2>0$
because $\dfrac{(2a-3c)^2}{9}+3c^2=0\implies c=0,a=0$ but $a\neq 0.$

Problem 10.6 (13-Met. Rec.)
$a(4a+2b+c)<0 \iff 4a^2+2ab+ac<0 \iff 16a^2+8ab<-4ac \iff$
$16a^2+8ab+b^2<b^2-4ac \iff (4a+b)^2<b^2-4ac \implies b^2-4ac>0.$

Problem 10.7(Met. Rec.)
For convenience, we write the function $f(x)$ given in the problem
in the form $f(x)=l_1(x)\,l_2(x)\ldots l_n(x)$,
where $l_i(x):=\dfrac{x-2i+1}{x-2i}=1+\dfrac{1}{x-2i}, i=1,2,...,n.$
Note that domain of $f(x)$ is $D(f)=\mathbb{R}\setminus\{2,4,...,2n\}$, where function $f(x)$ is differentiable.
For any $x\in D(f)$ we have $f'(x)=f(x)\sum\limits_{i=1}^{n}\dfrac{l_i'(x)}{l_i(x)}$.Since $l_i'(x)=-\dfrac{1}{x-2i}$ then
$f'(x)=-f(x)\sum\limits_{i=1}^{n}\dfrac{1}{(x-2i)(x-2i+1)}.$
Consider now two cases:
1. If $x<1$ or $x>2n$ then $f(x)>0$ and
$(x-2i)(x-2i+1)>0, i=1,2,...,n$ .Hence, $f'(x)<0$;
2. Let $x\in(2k-1,2k), k=1,2,..,n.$
Since $0>(x-2k)(x-2k+1)=(x-2k+1/2)^2-1/4\geq-1/4$
then $\dfrac{1}{(x-2k)(x-2k+1)}\leq-4$ with equality for $x=2k-1/2.$
From the other hand since $x\in(2k-1,2k)$ then
$x-2i>2k-1-2i=2(k-i)-1>0$ for $i<k$ and, therefore,
$(x-2i)(x-2i+1)>(2k-1-2i)(2k-1-2i+1)=2(k-i)(2(k-i)-1) \iff$

$$\frac{1}{(x-2i)(x-2i+1)} < \frac{1}{2(k-i)(2(k-i)-1)}.$$

Similarly, for $i > k$ we have

$$(x-2i)(x-2i+1) = (2i-x)(2i-x-1) > (2i-2k)(2i-2k-1) = 2(i-k)(2(i-k)-1) > 0$$

and, therefore, $\dfrac{1}{(x-2i)(x-2i+1)} < \dfrac{1}{2(i-k)(2(i-k)-1)}$.

Hence, if $k > 1$ then

$$\sum_{i=1}^{k-1} \frac{1}{(x-2i)(x-2i+1)} < \sum_{i=1}^{k-1} \frac{1}{2(k-i)(2(k-i)-1)} = \sum_{j=1}^{k-1} \frac{1}{2j(2j-1)} < \sum_{j=1}^{2k-1} \frac{1}{j(j+1)} < 1$$

and for $k = 1$ by definition of Summation Operator we have

$$\sum_{i=1}^{k-1} \frac{1}{(x-2i)(x-2i+1)} = 0$$

So, anyway $\displaystyle\sum_{i=1}^{k-1} \frac{1}{(x-2i)(x-2i+1)} < 1$ for any $k = 1, 2, ..., n$.

Similarly, $\displaystyle\sum_{i=k+1}^{n} \frac{1}{(x-2i)(x-2i+1)} < \sum_{i=k+1}^{n} \frac{1}{2(i-k)(2(i-k)-1)} =$

$$\frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + ... + \frac{1}{2(n-k)(2(n-k)-1)} < \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{(2n-1)2n} < 1\, if k < n.$$

If $k = n$ then by definition $\displaystyle\sum_{i=k+1}^{n} \frac{1}{(x-2i)(x-2i+1)} = 0$.

Thus, $\displaystyle\sum_{i=1}^{n} \frac{1}{(x-2i)(x-2i+1)} < 1 + (-4) + 1 = -2$.

Since $l_k(x) < 0$ and $l_i(x) > 0, i \neq k$ for $x \in (2k-1, 2k)$ then $f(x) < 0$ and, therefore, $f'(x) = -f(x) \displaystyle\sum_{i=1}^{n} \frac{1}{(x-2i)(x-2i+1)} < 0$.

Analysis.

Another way of solving this problem give opportunity to set and solve the
following generalization of the problem (SSMJ #5376).

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that $b_1 < a_1 < b_2 < a_2 < .... < a_{n-1} < b_n < a_n$.Let $F(x) := \dfrac{(x-b_1)(x-b_2)...(x-b_n)}{(x-a_1)(x-a_2)...(x-a_n)}$.

Prove that $F'(x) < 0$ for any $x \in Dom(F)$

Solution.

Lemma.

$F(x)$ can be represented in form

$$F(x) = 1 + \sum_{k=1}^{n} \frac{c_k}{x-a_k},$$

where $c_k, k = 1, 2, ..., n$ are some positive real numbers.

Proof.

Let $F_k(x) := \dfrac{(x-b_1)(x-b_2)...(x-b_k)}{(x-a_1)(x-a_2)...(x-a_k)}$, $k \le n$.

We will prove by Math. Induction that for any $k \le n$ there are positive numbers

$c_k(i), i = 1, ..., k$ such that $F_k(x) = 1 + \sum_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$.

Let $d_k := a_k - b_k > 0, k = 1, 2, ..., n$.

Note that $F_1(x) = \dfrac{x-b_1}{x-a_1} = \dfrac{x-a_1+a_1-b_1}{x-a_1} = 1 + \dfrac{d_1}{x-a_1}$.

Since $\dfrac{x-b_{k+1}}{x-a_{k+1}} = 1 + \dfrac{d_{k+1}}{x-a_{k+1}}$ then in supposition $F_k(x) = 1 + \sum_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$,

where $c_k(i) > 0, i = 1, ..., k < n$ we obtain

$$F_{k+1}(x) = F_k(x) \cdot \frac{x-b_{k+1}}{x-a_{k+1}} = \left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\right)\left(1 + \frac{d_{k+1}}{x-a_{k+1}}\right) =$$

$$1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} + \sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{(x-a_i)(x-a_{k+1})} = 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} -$$

$$\sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{a_{k+1}-a_i}\left(\frac{1}{x-a_i} - \frac{1}{x-a_{k+1}}\right) = 1 + \frac{d_{k+1}}{x-a_{k+1}}\left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{a_{k+1}-a_i}\right) +$$

$$\sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\left(1 - \frac{d_{k+1}}{a_{k+1}-a_i}\right) = 1 + \frac{d_{k+1}F_k(a_{k+1})}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} \cdot \frac{b_{k+1}-a_i}{a_{k+1}-a_i}.$$

Since $F_k(a_{k+1}) > 0$ and $b_{k+1} - a_i = (b_{k+1} - a_k) + (a_k - a_i) > 0$ then

$$c_{k+1}(k+1) = d_{k+1}F_k(a_{k+1}) > 0, \ c_{k+1}(i) := \frac{(b_{k+1}-a_i)\,c_k(i)}{a_{k+1}-a_i} > 0, i = 1, 2, ..., k$$

and $F_{k+1}(x) = 1 + \sum_{i=1}^{k+1} \dfrac{c_{k+1}(i)}{x-a_i}$.

Since $F(x) = 1 + \sum_{k=1}^{n} \dfrac{c_k}{x-a_k}$ and $c_k > 0, k = 1, 2, ..., n$ then

$F'(x) = -\sum_{k=1}^{n} \dfrac{c_k}{(x-a_k)^2} < 0$ for any $x \in Dom(F) = \mathbb{R} \setminus \{a_1, a_2, ..., a_n\}$.

Problem 10.8(20-Met. Rec.)

We will say that therms of the sequence $a_{i_1}, a_{i_2}, ..., a_{i_k}$ arranged in numerical order of $i_1 < i_2 < ... < i_k$ form the upper ladder (form the lower ladder) for a_m if holds two conditions:

1. $i_k = m$;

2. $a_{i_1} \ge a_{i_2} \ge ... \ge a_{i_k} (a_{i_1} \le a_{i_2} \le ... \le a_{i_k})$.

Wherein, k called "the height of the ladder".

Obvious that for any term of the sequence set of correspondent upper ladders (lower ladders) isn't empty. And besides, for any term of the sequence the height of its ladder bound by the number $n^2 + 1$.

Thus, for any term a_m of the sequence defined pair of nonnegative integer numbers (p_m, q_m) which, respectively, are the highest lower

ladder and the highest upper ladder for a_m.
Note that if $m_1 \neq m_2$ then $(p_{m_1}, q_{m_1}) \neq (p_{m_2}, q_{m_2})$.
Indeed, WLOG assume that $m_1 < m_2$. If $a_{m_1} \leq a_{m_2}$ then $p_{m_2} \geq p_{m_1} + 1$;
If $a_{m_1} \geq a_{m_2}$ then $q_{m_2} \geq q_{m_1} + 1$.So, we have exactly $n^2 + 1$ different pairs.
So, our problem can be formulated as follows:
Prove, that among $n^2 + 1$ numbers $a_1, a_2, ..., a_{n^2}, a_{n^2+1}$ there is at least one such that its ladder (no mettter upper or lower) has heigth not less then $n + 1$.
Assume contrary, that is for any $m \in \{1, 2, ..., n^2 + 1\}$ the heigth of any ladder for a_m does not exceed n.Then $p_m, q_m \in \{1, 2, ..., n\}$ and, therefore, total amount of pairs (p_m, q_m) does not exceed n^2.That is the contradiction.

Problem 10.9(22-Met. Rec.)

First note that for $m = 1$ inequality $\sqrt{7} - \dfrac{m}{n} > \dfrac{1}{mn}$ becomes

$$\sqrt{7} - \frac{1}{n} > \frac{1}{n} \iff \sqrt{7} > \frac{2}{n}$$

and obviously holds for any natural n.
We will prove that for any natural n, m such that $\dfrac{m}{n} < \sqrt{7}$ and $m \geq 2$ holds $7n^2 - m^2 \geq 3$.
Since $7n^2 - m^2 = 3$ for $n = 1$ and $m = 2$ and $\dfrac{2}{3} < \sqrt{7}$ then suffice to prove that equations $7n^2 - m^2 = 1, 7n^2 - m^2 = 2$ have no solutions in natural n, m such that $\dfrac{m}{n} < \sqrt{7}$.
Indeed,
$7n^2 - m^2 = 1 \implies m^2 \equiv -1 \pmod{7}$, $7n^2 - m^2 = 2 \implies m^2 \equiv 5 \pmod{7}$.
But for any integer m holds $m^2 \equiv 0, 1, 4, 2 \pmod{7}$.
Now we ready to complete the solution.
Suffice to note that $\sqrt{7} - \dfrac{m}{n} > \dfrac{1}{mn} \iff n\sqrt{7} - m > \dfrac{1}{m} \iff$
$n\sqrt{7} > m + \dfrac{1}{m} \iff 7n^2 > m^2 + 2 + \dfrac{1}{m^2} \iff$
$7n^2 - m^2 > 2 + \dfrac{1}{m^2}$ and $7n^2 - m^2 \geq 3 > 2 + \dfrac{1}{m^2}$.
By the way was proved that

$$\max\left\{m^2 - 7n^2 \mid m, n \in \mathbb{N} \text{ and } \frac{m}{n} < \sqrt{7}\right\} = -3.$$

Problem 10.10(39-Met. Rec.)

This problem has the following Interpretation:
Prove that $\min\left\{q \mid q \in \mathbb{N} \text{ and } \exists\,(p \in \mathbb{N}) \left[\dfrac{6}{13} < \dfrac{p}{q} < \dfrac{7}{15}\right]\right\} = 28.$
Note that for any fraction $\dfrac{a}{b}, \dfrac{c}{d}$ such that $\dfrac{a}{b} < \dfrac{c}{d}$ holds inequality

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Also note that if $bc - ad = 1$ *then* $\dfrac{a}{b}, \dfrac{c}{d}$ *both irreducible and since*

$(b+d)c-(a+c)d = bc-ad = 1$ *then* $\dfrac{a+c}{b+d}$ *is irreducible as well!*
In our problem $7\cdot 13 - 15\cdot 6 = 1$.
We generalize original problem in form of the following
Theorem.
Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be two positive fraction such that $\dfrac{a}{b} < \dfrac{c}{d}$ and $bc-ad=1$.
Then $\min\left\{q \mid q\in\mathbb{N} \text{ and } \exists\,(p\in\mathbb{N})\left[\dfrac{a}{b}<\dfrac{p}{q}<\dfrac{c}{d}\right]\right\} = b+d$.
Proof.
First note that $c(b+d)-d(a+c) = b(a+c)-a(b+d) = bc-ad = 1$.
Assume that there is a fraction $\dfrac{p}{q}$ such that $\dfrac{a}{b}<\dfrac{p}{q}<\dfrac{c}{d}$ and with $q<b+d$.
Since $\dfrac{a}{b}<\dfrac{p}{q} \implies pb-aq>0 \iff pb-aq\geq 1$ and

$\dfrac{p}{q}<\dfrac{c}{d} \implies qc-pd>0 \iff qc-pd\geq 1$

then $d(pb-aq)+b(qc-pd)\geq b+d \iff q(bc-ad)\geq b+d \iff q\geq b+d$.
Thus, we obtain the contradiction $q\leq b+d<b+d$ which complete the proof.■
Also we can see that for any fraction $\dfrac{p}{q}$ such that $\dfrac{a}{b}<\dfrac{p}{q}<\dfrac{c}{d}$ and $bc-ad=1$ holds $q\geq b+d$ and $p\geq a+c$ $(c(pb-aq)+a(qc-pd)\geq a+c \iff p(bc-ad)\geq a+c \iff p\geq a+c)$.
Let $\dfrac{p}{b+d}$ be fraction with minimal denumerator $b+d$ such that

$\dfrac{a}{b}<\dfrac{p}{b+d}<\dfrac{c}{d}$.

Assume that $p>a+c$. Since $0<c(b+d)-pd \iff 1\leq c(b+d)-pd$ then $1\leq c(b+d)-pd<c(b+d)-(a+c)d=bc-ad=1$ that is contradiction.
Therefore, $p=a+c$ and fraction with minimal denumerator defined uniquely and equal to $\dfrac{a+c}{b+d}$.
Remark.
In the case $0<\dfrac{a}{b},\dfrac{c}{d}$ such that $\dfrac{a}{b}<\dfrac{c}{d}$ and $bc-ad\neq 1$ the way of finding of "internal" fraction with minimal denumerator isn't works.

★Problem 10.11.
So, problem is:
Find all solution of equation

(1) $\overline{x_1x_2x_3...x_n} = \sum\limits_{i=1}^{n} x_i + \sum\limits_{1\leq i<j\leq n}^{n} x_ix_j + \sum\limits_{1\leq i<j<k\leq n}^{n} x_ix_jx_3 + ... + x_1x_2...x_n \iff$

$10^{n-1}x_1 + 10^{n-2}x_2 + ... + 10x_{n-1} + x_n + 1 = (1+x_1)(1+x_2)...(1+x_n)$,
where $x_1\in\{1,2,...,9\}$ and $x_2,x_3,...,x_n\in\{0,1,2,...,9\}$.

Lemma.
For any $x_1\in\{1,2,...,9\}$ and $x_2,x_3,...,x_n\in\{0,1,2,...,9\}$, $n\geq 2$
holds inequaliity

$(1+x_1)(1+x_2)...(1+x_n)\leq 10^{n-1}x_1+10^{n-2}x_2+...+10x_{n-1}+x_n+1$

and equality occurs iff $x_2 = x_3 = ... = x_n = 9$ and $x_1 \in \{1, 2, ..., 9\}$ be any.
Proof.(using Math Induction).
1. Base of Math Induction
Let $n = 2$ then we have $(1 + x_1)(1 + x_2) \le 10x_1 + x_2 + 1 \iff$
$1 + x_1 + x_2 + x_1x_2 \le 10x_1 + x_2 + 1 \iff x_1x_2 \le 9x_1 \iff x_1(9 - x_2) \ge 0.$
2. Step of Math Induction.
Let $x_1 \in \{1, 2, ..., 9\}$ and $x_2, x_3, ..., x_n, x_{n+1} \in \{0, 1, 2, ..., 9\}$.
If $x_{n+1} = 9$ then
$(1 + x_1)(1 + x_2)...(1 + x_n)(1 + x_{n+1}) \le$
$10^n x_1 + 10^{n-1}x_2 + 10^{n-2}x_2 + ... + 10x_n + x_{n+1} + 1 \iff$
$(1 + x_1)(1 + x_2)...(1 + x_n) \le 10^{n-1}x_1 + 10^{n-2}x_2 + ... + 10x_{n-1} + x_n + 1,$
where latter inequality holds by supposition of Math Induction
and equality occurs iff $x_2 = x_3 = ... = x_n = 9$,
$x_1 \in \{1, 2, ..., 9\}$ be any and $x_{n+1} = 9$.
Let now $x_{n+1} \in \{0, 1, 2, ..., 8\}$ .Then

$$\frac{10^n x_1 + 10^{n-1}x_2 + 10^{n-2}x_2 + ... + 10x_n + x_{n+1} + 1}{1 + x_{n+1}} >$$
$$1 + \frac{10^n x_1 + 10^{n-1}x_2 + 10^{n-2}x_2 + ... + 10x_n}{10} =$$
$$10^{n-1}x_1 + 10^{n-2}x_2 + ... + 10x_{n-1} + x_n + 1.$$

By supposition of Math Induction we have
$10^{n-1}x_1 + 10^{n-2}x_2 + ... + 10x_{n-1} + x_n + 1 \ge (1 + x_1)(1 + x_2)...(1 + x_n)$.
Hence, $10^n x_1 + 10^{n-1}x_2 + 10^{n-2}x_2 + ... + 10x_n + x_{n+1} + 1 \ge$
$(1 + x_1)(1 + x_2)...(1 + x_n)(1 + x_{n+1})$ ■.
Using Lemma we immediatelly obtain that all solutions of equation **(1)** are numbers $\overline{x_1 99...9}$ and $x_1 \in \{1, 2, ..., 9\}$.

Problem 10.12(51-Met. Rec.).
Let n be an integer root of $P(x)$ then $\dfrac{P(n) - P(0)}{n - 0} = -\dfrac{P(0)}{n} \in \mathbb{Z} \implies n$ is odd as divisor of odd number.
From another hand $\dfrac{P(n) - P(1)}{n - 1} = -\dfrac{P(1)}{n - 1} \in \mathbb{Z} \implies n - 1$ is odd as divisor of odd number.
But since n odd then $n - 1$ is even. So x os odd and even simulteneously-that is contradiction.

Problem 13(52-Met. Rec.).

Let a, b, c be different integer numbers such that $P(a) = P(b) = P(c) = 1$ and assume that n is integer root of $P(x)$. Then for $x \in \{a, b, c\}$ we have
$\dfrac{P(n) - P(x)}{n - x} = \dfrac{-1}{n - x} \in \mathbb{Z} \implies n - x \in \{1, -1\}$.
Since three numbers $n - a, n - b, n - c$ belong to 2-elements set then at least two of them is equal.

But that yields that equal two of a, b, c and it is contradiction.

Problem 10.14(53-Met. Rec.).
We will prove using Math Induction that $P(n+km)$ divisible by m for any natural k.
For $k=1$ we have
$\dfrac{P(n+m)-P(n)}{(n+m)-n} = \dfrac{P(n+m)-m}{m} \in \mathbb{Z} \iff \dfrac{P(n+m)}{m} \in \mathbb{Z}.$
For any natural k assuming that $\dfrac{P(n+km)}{m} \in \mathbb{Z}$ we obtain
$\dfrac{P(n+(k+1)m)-P(n+km)}{(n+(k+1)m)-(n+km)} = \dfrac{P(n+(k+1)m)-P(n+km)}{m} \in \mathbb{Z} \implies \dfrac{P(n+(k+1)m)}{m}.$

Problem 10.15(54-Met. Rec.).

Let $x_0 := x$ and $x_k := f(x_{k-1}), k \in \mathbb{N}$ then $g(x) = \underbrace{f(f(...f(x)...))}_{n-\text{times}} = x_n$.

a) $x_1 = \dfrac{x}{\sqrt{1-x^2}},\ x_2 = \dfrac{x_1}{\sqrt{1-x_1^2}} = \ x_2 = \dfrac{\dfrac{x}{\sqrt{1-x^2}}}{\sqrt{1-\dfrac{x^2}{1-x^2}}} = \dfrac{x}{\sqrt{1-2x^2}}.$

Assume that $x_k = \dfrac{x}{\sqrt{1-kx^2}}$ then $x_{k+1} = \dfrac{x_k}{\sqrt{1-x_k^2}} = \dfrac{\dfrac{x}{\sqrt{1-kx^2}}}{\sqrt{1-\dfrac{x^2}{1-kx^2}}} =$
$\dfrac{x}{\sqrt{1-(k+1)x^2}}.$
So, byMath Induction we proved that $x_k = \dfrac{x}{\sqrt{1-kx^2}}$ for any $k \in \mathbb{N}$ and, therefore, $g(x) = x_n = \dfrac{x}{\sqrt{1-nx^2}}$

b) Note that $f(\cot t) = \dfrac{\cot t\sqrt{3}-1}{\cot t+\sqrt{3}} = \dfrac{\cot t \cot \dfrac{\pi}{6} - 1}{\cot t + \cot \dfrac{\pi}{6}} = \cot\left(t + \dfrac{\pi}{6}\right)$
for any $t \in \mathbb{R}$.
Let $\varphi_0 := \cot^{-1}(x)$ and $\varphi_k = \varphi_{k-1} + \dfrac{\pi}{6}, k \in \mathbb{N}$ then $\varphi_k = \varphi_0 + \dfrac{k\pi}{6}, k \in \mathbb{N}$.
We will prove by Math Induction that $x_k = \cot\varphi_k, k \in \mathbb{N}\cup\{0\}$.
Base of Math Induction.
We have by definition $x_0 = x = \cot\varphi_0$.
Step of Math Induction.
For any $k \in \mathbb{N}\cup\{0\}$ suppposition $x_k = \cot\varphi_k$ yeilds
$x_{k+1} = f(x_k) = f(\cot\varphi_k) = \cot\varphi_{k+1}$.

Thus, $g(x) = x_n = \cot\varphi_n = \cot\left(\varphi_0 + \frac{n\pi}{6}\right) = \frac{\cot\varphi_0 \cot\frac{n\pi}{6} - 1}{\cot\varphi_0 + \cot\frac{n\pi}{6}} = \frac{x\cot\frac{n\pi}{6} - 1}{x + \cot\frac{n\pi}{6}}$

Problem 10.16(62-Met. Rec.).

Note that since $F(x) = \frac{4^x}{4^x + 2} = \frac{2^x}{2^x + 2^{1-x}}$ then

$F(1-x) = \frac{2^{1-x}}{2^x + 2^{1-x}}$ and $F(x) + F(1-x) = 1$.

Let $S_n := \sum_{k=0}^{n} F\left(\frac{k}{n}\right)$. Then $S_n = \sum_{k=0}^{n} F\left(\frac{n-k}{n}\right)$ and, therefore,

$2S_n := \sum_{k=0}^{n} F\left(\frac{k}{n}\right) + \sum_{k=0}^{n} F\left(\frac{n-k}{n}\right) =$

$\sum_{k=0}^{n}\left(F\left(\frac{k}{n}\right) + F\left(1-\frac{k}{n}\right)\right) = n+1$.Hence, $S_n = \frac{n+1}{2}$ and

$\sum_{k=1}^{n} F\left(\frac{k}{n}\right) = S_n - F(0) = \frac{n+1}{2} - \frac{1}{3} = \frac{3n+1}{6}$.

Problem 10.17(63-Met. Rec.).

Let $q_1 < q_2$ and $x_i := f(q_i), i = 1, 2$. Then, since $x_i^3 + px_i - q_i = 0, i = 1, 2$ we obtain $x_2^3 + px_2 - q_2 - \left(x_1^3 + px_1 - q_1\right) = 0 \iff$
$x_2^3 - x_1^3 + p(x_2 - x_1) = q_2 - q_1 \iff$
$(x_2 - x_1)\left(x_2^2 + x_2x_1 + x_1^2 + p\right) = q_2 - q_1 \iff x_2 - x_1 = \frac{q_2 - q_1}{x_2^2 + x_2x_1 + x_1^2 + p}$.
Hence, $x_2 - x_1 > 0$ because $q_2 - q_1 > 0$ and $x_2^2 + x_2x_1 + x_1^2 + p \geq p > 0$ for any x_1, x_2 and, therefore, $f(q_2) > f(q_1)$.

Problem 10.18(64-Met. Rec.).

Assume that there is $a \in \mathbb{R}$ such that $P(P(a)) = a$ then
$P(P(a)) - a = 0 \iff P(P(a)) - P(a) + P(a) - a = 0 \iff$
$P(P(a)) - P(a) = -(P(a) - a)$.
Let $b := P(a)$ and $f(x) := P(x) - x$.Since $P(a) \neq a \iff b \neq a$ and
$P(P(a)) - P(a) = -(P(a) - a) \iff P(b) - b = -(P(a) - a) \iff$
$f(b) = -f(a)$ then $f(a) f(b) < 0$ and $f(x)$ as continuous function has a root ■ located between a and b,that is $f(c) = 0 \iff P(c) = c$.
Obtained contradiction mean that equation $P(P(x)) = x$ have no roots as well.

Problem 10.19(67-Met. Rec.).

According to the statement of the problem we have two sequences of numbers $a_1, a_2, ..., a_n$ (boys) and $b_1, b_2, ..., b_n$ (girls) for which one of the two conditions holds:
a) $a_i < b_i, i = 1, 2, ..., n$ or **b)** $|a_i - b_i| < h, i = 1, 2, ..., n$.

(in the problem $h = 10$)

1. Since conditions **a)** and **b)** connect in pairs only terms of both sequences, that standing on the places with the same numbers, then the fulfillment of these conditions does not depend on the order of order listing of these pairs.

Therefore, without loss of generality, we may assume that the members of one of the two sequences, let it be $b_1, b_2, ..., b_n$ is originally ordered as $b_1 \le b_2 \le ... \le b_n$ and then should be ordered only the sequence $a_1, a_2, ..., a_n$.

Since the ordering of any sequence of numbers is reduced to the implementation of ordering transpositions, that is two terms of the sequence a_i and $a_j, i < j$ exchanged their positions if $a_i > a_j$, it is suffices to prove the invariance of fulfilling of the properties **a)** and **b)** when the corresponding transposition is made. That is, to prove the validity of the proposed claims in the case of $n = 2$ for pairs (a_1, a_2) and (b_1, b_2) in the supposition that $b_1 \le b_2$.

Suppose that $a_1 > a_2$. In the case of **a)** we have $a_1 \le b_1, a_2 \le b_2$ and $b_1 \le b_2$.

After transposition we obtain pair (a_2, a_1). Then $a_2 < a_1 \le b_1 \le b_2$ yields $a_2 < b_1$ and $a_1 \le b_2$;

In the case of **b)** we have $|a_1 - b_1| \le h, |a_2 - b_2| \le h$ and $a_1 > a_2$.

We will prove $|a_2 - b_1| \le h, |a_1 - b_2| \le h$.

Indeed, since $|a_1 - b_1| \le h \iff b_1 - h \le a_1 \le b_1 + h$ and

$|a_2 - b_2| \le h \iff b_2 - h \le a_2 \le b_2 + h$

then $\begin{cases} a_1 \le b_1 + h \\ b_1 \le b_2 \end{cases} \implies a_1 \le b_2 + h$ and

$\begin{cases} b_2 - h \le a_2 \\ a_2 < a_1 \end{cases} \implies b_2 - h < a_1$.

Hence, $|a_1 - b_2| \le h$.

Similarly, $\begin{cases} a_1 \le b_1 + h \\ a_2 < a_1 \end{cases} \implies a_2 < b_1 + h$ and

$\begin{cases} b_2 - h \le a_2 \\ b_1 \le b_2 \end{cases} \implies b_1 - h < a_1$.

Hence, $|a_2 - b_1| \le h$.

Problem 10.20 (86-Met. Rec).

Solution1.

Denoting $u := x - b, v := y - b$ we set free parameter b and obtain system of inequalities that equivalent to original system, namely the system

(1) $\begin{cases} u + b \ge v^2 \\ v + b \ge u^2 \end{cases} \iff \begin{cases} u \ge v^2 - b \\ v \ge u^2 - b \end{cases}$.

Let (u, v) be only sollution of the system **(1).**

Then $u + v \ge v^2 - b + u^2 - b \iff$

$$2b + \frac{1}{2} \ge \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 \ge 0 \implies b \ge -\frac{1}{4}.$$

If $b > -\dfrac{1}{4}$ then system **(1)** have at least two solutions**.**

Indeed**,** in that case equation $b = z^2 - z \iff z^2 - z - b = 0$ have

two roots $z_1 = \dfrac{1-\sqrt{1+4b}}{2}$ and $z_2 = \dfrac{1+\sqrt{1+4b}}{2}$
and, therefore, $(u,v) = (z_1, z_1), (z_2, z_2)$ are two different solution of the system **(1).**
Thus, can't be $b > -\dfrac{1}{4}$ and remains $b = -\dfrac{1}{4}$ as necessary condition for b to provide only solution of the system **(1).**
And, vise versa, if $b = -\dfrac{1}{4}$ then

$$\begin{cases} u - 1/4 \ge v^2 \\ v - 1/4 \ge u^2 \end{cases} \implies \left(u-\frac{1}{2}\right)^2 + \left(v-\frac{1}{2}\right)^2 = 0 \iff (u,v) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and it is only solution of the system.

Solution2. (Direct solution of original problem).
Let b be such that original system has only solution (x,y).
Note that $x = y$ because if $x \ne y$ then due symmetry of the system pair (y,x) which not equal to (x,y) is solution as well and that contradicts to uniqueness of solution (x,y).
But if $x = y$ then system of two inequalities becomes one inequality

$$x \ge (x-b)^2 \iff x^2 - (2b+1)x + b^2 \le 0$$

and this inequality has unique solution. That possible only iff discriminant of equation $x^2 - (2b+1)x + b^2 = 0$ equal to zero, that is iff

$$(2b+1)^2 - 4b^2 = 0 \iff b = -\frac{1}{4}.$$

Now, let $b = -\dfrac{1}{4}$ then $\begin{cases} x \ge \left(y+\dfrac{1}{4}\right)^2 \\ y \ge \left(x+\dfrac{1}{4}\right)^2 \end{cases} \implies$

$$x + y \ge \left(y+\frac{1}{4}\right)^2 + \left(x+\frac{1}{4}\right)^2 \iff$$

$$\left(y-\frac{1}{4}\right)^2 + \left(x-\frac{1}{4}\right)^2 \le 0 \iff (x,y) = (1/4, 1/4).$$

Remark.
Note that $\begin{cases} u + b \ge v^2 \\ v + b \ge u^2 \end{cases} \iff \begin{cases} b \ge v^2 - u \\ b \ge u^2 - v \end{cases} \iff$

$$b \ge \max\left\{u^2 - v, v^2 - u\right\} \text{ and } b = -1/4$$

which provide uniqueness of solution at the same time is

$$\min_{u,v\in\mathbb{R}} \left(\max\left\{u^2 - v, v^2 - u\right\}\right).$$

Indeed, $\max\left\{u^2 - v, v^2 - u\right\} \ge \dfrac{u^2 - v + v^2 - u}{2} = \dfrac{\left(u^2 - u\right) + \left(v^2 - v\right)}{2} = \dfrac{(u-1/2)^2 + (v-1/2) - 1/2}{2} \ge -1/4$ and since lower bound for $\max\left\{u^2 - v, v^2 - u\right\}$ is attanable if $(u,v) = (1/2, 1/2)$ then

$$\min_{u,v\in\mathbb{R}} \left(\max\left\{u^2 - v, v^2 - u\right\}\right) = -1/4.$$

★Problem 10.21(CRUX 3090)
Suppose $x=\min\{x,y,z\}$, then
$3-4y\le 3-4x \implies 2x(3-4x)\ge 2x(3-4y)\ge z^2+1\ge x^2+1.$
So, $2x(3-4x)\ge x^2+1 \iff 9x^2-6x+1\le 0 \iff x=\dfrac{1}{3}$,
and because $x=\min\{x,y,z\}$ that implies $y\ge\dfrac{1}{3}$ and $z\ge\dfrac{1}{3}$.
From other side

$$x=\frac{1}{3}\implies z^2+1\le\frac{2}{3}(3-4y)\le\frac{2}{3}\left(3-\frac{4}{3}\right)=\frac{10}{9}\implies z^2\le\frac{1}{9}\implies z\le\frac{1}{3}.$$

So, $z=\dfrac{1}{3}$. The same way gives us $y=\dfrac{1}{3}$.■

★Problem 10.22(87-Met. Rec).
Let x_i be number of $2-$rings chains created from rings taken by one from rods staing in the points A_i and $A_{i+1}, i=1,2,3,4$ $(A_5=A_1)$.
Then we should maximaze sum $x_1+x_2+x_3+x_4$
by all quads (x_1,x_2,x_3,x_4) of nonnegative integer numbers such that
$x_1+x_4\le a_1, x_1+x_2\le a_2, x_2+x_3\le a_3, x_3+x_4\le a_4.$
Set of all such quads (x_1,x_2,x_3,x_4) we denote D.
So, we have to determine $\max\limits_{(x_1,x_2,x_3,x_4)\in D}(x_1+x_2+x_3+x_4)$.
Let $t:=x_4$ then $0\le t\le\min\{a_1,a_4\}$ and
$D_t:=\{(x_1,x_2,x_3)\mid x_i\ge 0, i=1,2,3, x_1\le a_1-t, x_1+x_2\le a_2, x_2+x_3\le a_3, x_3\le a_4-t\}$.
Thus, $\max\limits_{(x_1,x_2,x_3,x_4)\in D}(x_1+x_2+x_3+x_4)=\max\limits_{0\le t\le\min\{a_1,a_4\}}\left(t+\max\limits_{(x_1,x_2,x_3)\in D_t}(x_1+x_2+x_3)\right)$.

Lemma.
Let $x_1\le b_1, x_1+x_2\le b_2, x_2+x_3\le b_3, x_3\le b_4, x_i\ge 0, i=1,2,3$ where $b_i,\ i=1,2,3$ are given nonnegative integer numbers.
Then maximal possible value of $x_1+x_2+x_3$ equal
$\min\{b_1+b_3, b_2+b_3, b_2+b_4\}$.

Proof.
We have $\begin{cases} x_1\le b_1\\ x_1+x_2\le b_2\\ x_2+x_3\le b_3\\ x_3\le b_4\\ 0\le x_1,x_2,x_3\end{cases} \iff \begin{cases} 0\le x_1\le b_1\\ 0\le x_2\le b_2-x_1\\ 0\le x_3\le b_3-x_2\\ x_3\le b_4\\ x_1\le b_2\\ x_2\le b_3\end{cases} \iff \begin{cases} 0\le x_1\le\min\{b_1,b_2\}\\ 0\le x_2\le\min\{b_2-x_1,b_3\}\\ 0\le x_3\le\min\{b_3-x_2,b_4\}\end{cases}$.
Then $x_1+x_2+x_3\le x_1+x_2+\min\{b_3-x_2,b_4\}=x_1+\min\{b_3,x_2+b_4\}\le$
$x_1+\min\{b_3,\min\{b_2-x_1,b_3\}+b_4\}=x_1+\min\{b_3,b_2-x_1+b_4,b_3+b_4\}=$
$x_1+\min\{b_3,b_2+b_4-x_1\}=\min\{b_3+x_1,b_2+b_4\}\le\min\{b_3+\min\{b_1,b_2\},b_2+b_4\}=$
$\min\{b_1+b_3,b_2+b_3,b_2+b_4\}$. Let $x_1^*:=\min\{b_1,b_2\}, x_2^*:=\min\{b_2-x_1^*,b_3\}$,
$x_3^*:=\min\{b_3-x_2^*,b_4\}$.
Then $x_1+x_2+x_3\le x_1^*+x_2^*+x_3^*=\min\{b_1+b_3,b_2+b_3,b_2+b_4\}$ and,
therefore, $\max(x_1+x_2+x_3)=\min\{b_1+b_3,b_2+b_3,b_2+b_4\}$.

By replacing (b_1, b_2, b_3, b_4) in Lemma with $(a_1 - t, a_2, a_3, a_4 - t)$ we obtain that

$\max\limits_{(x_1,x_2,x_3)\in D_t} (x_1 + x_2 + x_3) = \min\{a_1 - t + a_3, a_2 + a_3, a_2 + a_4 - t\}$

and, therefore,

$\max\limits_{(x_1,x_2,x_3,x_4)\in D} (x_1 + x_2 + x_3 + x_4) = \max\limits_{0\le t\le \min\{a_1,a_4\}} (t + \min\{a_1 - t + a_3, a_2 + a_3, a_2 + a_4 - t\}) =$
$\max\limits_{0\le t\le \min\{a_1,a_4\}} (\min\{a_1 + a_3, a_2 + a_3 + t, a_2 + a_4\}) = \min\{a_1 + a_3, a_2 + a_3 + \min\{a_1, a_4\}, a_2 + a_4\} =$
$\min\{a_1 + a_3, a_2 + a_3 + a_1, a_2 + a_3 + a_4, a_2 + a_4\} = \min\{a_1 + a_3, a_2 + a_4\}.$

Problem 10.23(Problem with light bulbs).

States of bulbs is encoded by two numbers -1 if bulb is turned on and 1 if it turned off.

For any integer number m let $D(m)$ be set of all natural divisors of m.

Let $a_m(k)$, $m = 1, 2, ..., n$ be state of the $m - th$ bulb when the person click $k - th$ switch.

Note that $a_m(1) = 1$ for all $m \in \{1, 2, ...n\}$ and

$$a_m(k) = \begin{cases} a_m(k-1) & \text{if } k \notin D(m) \\ -a_m(k-1) & \text{if } k \in D(m) \end{cases}, k \in \{2, 3, ..., n\}, m \in \{1, 2, ...n\}$$

Since $a_m(n) = (-1)^{|D(m)|}, m \in \{1, 2, ...n\}$ then $a_m(n)$ is turned on iff $|D(m)|$ is odd number.

If at least one of exponent in expansion $m = p_1^{\alpha_1}...p_l^{\alpha_l}$ is odd then $|D(m)|$ is even.

Thus, $|D(m)|$ is odd iff all expoents are even, that is iff m is a perfect square and, therefore, we have so many turned on bulbs as haw many perfect squares between 1 and n.

Since $1 \le k^2 \le n \iff 1 \le k \le [\sqrt{n}]$ answer is:

$[\sqrt{n}]$ bulbs finally will be turned on.

Problem 10.24(O274, MR4 2013).

We have $\begin{cases} bx + ay \le abc \\ x, y \ge 0. \end{cases} \iff \begin{cases} 0 \le y \le \left\lfloor \dfrac{b(ac - x)}{a} \right\rfloor \\ 0 \le x \le ac. \end{cases} \iff$

$$\begin{cases} 0 \le t \le ac \\ x = ac - t \\ 0 \le y \le \left\lfloor \dfrac{bt}{a} \right\rfloor. \end{cases}.$$

Hence, $D := \left\{(ac - t, y) \mid 0 \le t \le ac \text{ and } 0 \le y \le \left\lfloor \dfrac{bt}{a} \right\rfloor\right\}$ and

$|D| = \sum\limits_{t=0}^{ac} \left(\left\lfloor \dfrac{bt}{a} \right\rfloor + 1\right) = ac + 1 + \sum\limits_{t=0}^{ac} \left\lfloor \dfrac{bt}{a} \right\rfloor$.Since $\{0, 1, 2, ..., ac\} =$
$\{ac\} \cup \{ka + r \mid k = 0, 1, ..., c - 1 \text{ and } r = 0, 1, 2, ..., a - 1\}$ then

$\sum\limits_{t=0}^{ac} \left\lfloor \dfrac{bt}{a} \right\rfloor = bc + \sum\limits_{k=0}^{c-1}\sum\limits_{r=0}^{a-1} \left\lfloor \dfrac{b(ka + r)}{a} \right\rfloor = bc + \sum\limits_{k=0}^{c-1}\sum\limits_{r=0}^{a-1} \left(bk + \left\lfloor \dfrac{br}{a} \right\rfloor\right) =$

$bc+\sum_{k=0}^{c-1}\sum_{r=0}^{a-1}bk+\sum_{k=0}^{c-1}\sum_{r=0}^{a-1}\left\lfloor\frac{br}{a}\right\rfloor=bc+ab\sum_{k=0}^{c-1}k+\sum_{r=0}^{a-1}\sum_{k=0}^{c-1}\left\lfloor\frac{br}{a}\right\rfloor=$
$bc+\frac{ab\left(c-1\right)c}{2}+c\sum_{r=1}^{a-1}\left\lfloor\frac{br}{a}\right\rfloor.$
Since $\sum_{r=1}^{a-1}\left\lfloor\frac{br}{a}\right\rfloor=\sum_{r=1}^{a-1}\left(\frac{br}{a}-\left\{\frac{br}{a}\right\}\right)=\sum_{r=1}^{a-1}\frac{br}{a}-\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}=$
$\frac{b}{a}\sum_{r=1}^{a-1}r-\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}=\frac{ba\left(a-1\right)}{2a}-\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}=\frac{b\left(a-1\right)}{2}-\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}$
then remaince evaluate sum $\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}$.
Since $a\perp b$ then $\left\{a\left\{\frac{br}{a}\right\}\mid\ r=1,2,...,a-1\right\}=\{1,2,...,a-1\}$ then
$\sum_{r=1}^{a-1}\left\{\frac{br}{a}\right\}=\sum_{r=1}^{a-1}\frac{r}{a}=\frac{a-1}{2}$ and, therefore,
$\sum_{r=1}^{a-1}\left\lfloor\frac{br}{a}\right\rfloor=\frac{b\left(a-1\right)}{2}-\frac{a-1}{2}=\frac{\left(a-1\right)\left(b-1\right)}{2}.$
Thus, $|D|=ac+1+bc+\frac{ab\left(c-1\right)c}{2}+\frac{c\left(a-1\right)\left(b-1\right)}{2}=$
$\frac{c\left(2a+2b+abc-ab+ab-a-b+1\right)}{2}+1=\frac{c\left(abc+a+b+1\right)}{2}+1.$
.

Problem 10.25(102-Met. Rec.)
a) Suppose that $\sin x+\sin\alpha x$ is periodic with the period τ.
Then $\sin\left(x+\tau\right)+\sin\alpha\left(x+\tau\right)=\sin x+\sin\alpha x\iff$
$\sin\left(x+\tau\right)-\sin x=-\left(\sin\alpha\left(x+\tau\right)-\sin\alpha x\right).$
Let $h\left(x\right):=\sin\left(x+\tau\right)-\sin x=-\left(\sin\alpha\left(x+\tau\right)-\sin\alpha x\right).$
Then $h\left(x\right)$ is periodic with period τ.But at the same time $h\left(x\right)$ have periods 2π and $\frac{2\pi}{\alpha}$.
Note that $\tau\notin2\pi\mathbb{Z}\setminus\{0\}$ because otherwise if $\tau\in2\pi\mathbb{Z}\setminus\{0\}$ then
$\sin\alpha\left(x+\tau\right)-\sin\alpha x=0$ for any x and in paricular if $x=0$.
Then $\sin\alpha\tau=0\iff\alpha\tau=k\pi\implies\alpha=\frac{k\pi}{\tau}\in\mathbb{Q}$ −contradiction.
Since continuos function $h\left(x\right)$ isn't constant then it has smallest positive period τ_*.Then $2\pi=k\tau_*$ and $\frac{2\pi}{\alpha}=l\tau_*$ for some integer k,l and, therefore, $\alpha=\frac{k}{l}\in\mathbb{Q}$.
Contradiction!

Another solution.
Let $h\left(x\right):=\sin x+\sin\alpha x$, then $h'\left(x\right)=\cos x+\alpha\cos\alpha x$ and $h''\left(x\right)=-\sin x-\alpha^2\sin\alpha x$
Assume that $h\left(x\right)$ is periodic with period τ.
Then $h'\left(x\right)$ and $h''\left(x\right)$ are periodic with period τ.
Since $h\left(x\right)+h''\left(x\right)=\left(1-\alpha^2\right)\sin\alpha x$ and $\sin\alpha x$ has main period $\frac{2\pi}{\alpha}$

then $\tau = \dfrac{2\pi}{\alpha} m$ because $h'(x) + h''(x)$ has period τ.
Similarly, since $\alpha^2 h(x) + h''(x) = (\alpha^2 - 1)\sin x$ we obtain
$\tau = 2n\pi$.Hence, $\dfrac{2\pi}{\alpha} m = 2n\pi \iff \alpha = \dfrac{m}{n} \in \mathbb{Q}$,that is contradiction.
b) Solution similar to **a)**
c) Let $h(x) := \tan x + \tan \alpha x$ where $\alpha \notin \mathbb{Q}$.
Then $h'(x) = 1 + \tan^2 x + \alpha\left(1 + \tan^2 \alpha x\right) = 1 + \alpha + \tan^2 x + \alpha \tan^2 \alpha x$,
$\tan\tau + \tan\alpha\tau = 0, \tan^2\tau + \alpha\tan^2\alpha\tau = 0$,
$\tan\tau\left(1 + \tan^2\tau\right) + 2\alpha^2\tan\alpha\tau\left(1 + \tan^2\alpha\tau\right) = 0$.
Since $\tan\alpha\tau = -\tan\tau$ then $\tan^2\tau + \alpha\tan^2\alpha\tau = \tan^2\tau(1+\alpha) = 0 \iff$
$\tau = n\pi, n \in \mathbb{Z}$ and $\tan\alpha\tau = 0 \iff \alpha\tau = m\pi, m \in \mathbb{Z}$.
Hence, $\alpha = \dfrac{m}{n} \in \mathbb{Q}$ and that is contradiction.
More simple.
Since $\cot x + \cot \alpha x$ isn't periodic (because $D(\cot x + \cot\alpha x) =$
$D(\cot x) \cap D(\cot\alpha x) = \mathbb{R}\diagdown\pi\mathbb{Z} \cap \mathbb{R}\diagdown\dfrac{\pi}{\alpha}\mathbb{Z}$ is nonperiodic) then
$\tan x + \tan\alpha x = \cot(\pi/2x - x) + \cot(\pi/2x - \alpha x)$
is nonperiodic as well.
d) Let $h(x) := \sin x + \tan\alpha x$ where $\alpha \notin \mathbb{Q}$. Suppose that $h(x)$ periodic with period $\tau > 0$.
Since $h(x)$ is differentiable then
$h'(x) = \cos x + \alpha\left(1 + \tan^2\alpha x\right), h''(x) = -\sin x + 2\alpha^2\tan\alpha x\left(1 + \tan^2\alpha x\right)$
are periodic with period τ as well. Since $h(\tau) = h(0), h'(\tau) = h'(0)$,
$h''(\tau) = h''(0)$ then τ satisfy to the system.
Hence

$$\begin{cases} \sin\tau + \tan\alpha\tau = 0 \\ \cos\tau + \alpha\left(1 + \tan^2\alpha\tau\right) = 1 + \alpha \\ -\sin\tau + 2\alpha^2\tan\alpha\tau\left(1 + \tan^2\alpha\tau\right) = 0 \end{cases} \iff \begin{cases} \sin\tau + \tan\alpha\tau = 0 \\ \cos\tau + \alpha\tan^2\alpha\tau = 1 \\ -\sin\tau + 2\alpha^2\tan\alpha\tau\left(1 + \tan^2\alpha\tau\right) = 0 \end{cases}.$$

Then $\sin\tau + \tan\alpha\tau + (-\sin\tau) + 2\alpha^2\tan\alpha\tau\left(1 + \tan^2\alpha\tau\right) = 0 \iff \tan\alpha\tau\left(1 + 2\alpha^2\left(1 + \tan^2\alpha\tau\right)\right) = 0 \iff$
$\tan\alpha\tau = 0 \iff \alpha\tau = n\pi, n \in \mathbb{Z}$ and, therefore, $\sin\tau = 0 \iff \tau = m\pi, m \in \mathbb{Z}$
Hence, $n\pi = m\pi\alpha \iff \alpha = \dfrac{n}{m} \in \mathbb{Q}$ and that is contradiction.

Problem 10.26(103-Met.Rec)

Denote $S_n = \dfrac{1}{a_1 + 1} + \dfrac{1}{a_2 + 1} + ... + \dfrac{1}{a_n + 1}$.
Since $x_{k+1} = x_k + x_k^2$ then from $\dfrac{1}{x_{k+1}} = \dfrac{1}{x_k(1 + x_k)} = \dfrac{1}{x_k} - \dfrac{1}{x_k + 1} =$
$\dfrac{1}{x_k} - \dfrac{x_k}{x_k(x_k + 1)} = \dfrac{1}{x_k} - \dfrac{x_k}{x_{k+1}}$ follows that $\dfrac{x_k}{x_{k+1}} = \dfrac{1}{x_k} - \dfrac{1}{x_{k+1}}$.
Thus, we obtain $S_n = \dfrac{1}{x_1} - \dfrac{1}{x_{n+1}}$.
Or more shortly:
Dividing $a_{n+1} = a_n + a_n^2$ by $a_n a_{n+1}$ we obtain

$\dfrac{1}{a_n}-\dfrac{1}{a_{n+1}}=\dfrac{a_n}{a_{n+1}}=\dfrac{1}{a_n+1}$ and from that immediately follows

$S_n=\sum\limits_{k=1}^{n}\left(\dfrac{1}{a_k}-\dfrac{1}{a_{k+1}}\right)=\dfrac{1}{a_1}-\dfrac{1}{a_{n+1}}=2-\dfrac{1}{a_{n+1}}.$

So, $S_n<2$ for any $n\in\mathbb{N}$ From the other hand, since a_n increasing in $\mathbb{N}$ (this follows from $a_{n+1}-a_n=a_n^2>0, n\in\mathbb{N}$) then

$a_n\geq a_3=\dfrac{3}{4}+\dfrac{9}{16}=\dfrac{21}{16}>1$ for all $n\geq 3$.

Hence, $S_n=2-\dfrac{1}{a_{n+1}}\geq 2-\dfrac{1}{a_3}>1$ for any $n\geq 2$.

Thus for all $n\geq 3$ holds $1<S_n<2 \iff [S_n]=1$.

(Or, alternatively, since $a_2=\dfrac{3}{4}$ then $S_2=\dfrac{1}{1+\dfrac{1}{2}}+\dfrac{1}{1+\dfrac{3}{4}}=\dfrac{2}{3}+\dfrac{4}{7}=\dfrac{26}{21}>1$

and $1<S_2\leq S_n<2$ for any $n\geq 2$).

Problem 10.27 (Austria – Poland, 1980).

Since $|a_{n+m}-a_n-a_m|\leq 1 \iff a_n+a_m-1\leq a_{n+m}\leq a_n+a_m+1$ then in particularly

$2a_n-1\leq a_{2n}\leq 2a_n+1,\ 3a_n-2\leq 2a_n+a_n-1\leq a_{2n}\leq a_{2n}+a_n+1\leq 3a_n+2$

and further, using math induction we obtain

$ma_n-(m-1)\leq a_{mn}\leq ma_n+(m-1)$.

Hence $\dfrac{a_n}{n}-\dfrac{m-1}{mn}\leq\dfrac{a_{mn}}{mn}\leq\dfrac{a_n}{n}+\dfrac{m-1}{mn} \iff \left|\dfrac{a_{mn}}{mn}-\dfrac{a_n}{n}\right|\leq\dfrac{m-1}{mn}$

and since

$\dfrac{m-1}{mn}<\dfrac{1}{n}$ then $\left|\dfrac{a_{mn}}{mn}-\dfrac{a_n}{n}\right|<\dfrac{1}{n}$.

Switching places for n and m we get also $\left|\dfrac{a_{mn}}{mn}-\dfrac{a_m}{m}\right|<\dfrac{1}{m}$.

Hereof, $\left|\dfrac{a_n}{n}-\dfrac{a_m}{m}\right|=\left|\dfrac{a_n}{n}-\dfrac{a_{mn}}{mn}+\dfrac{a_{mn}}{mn}-\dfrac{a_m}{m}\right|\leq\left|\dfrac{a_{mn}}{mn}-\dfrac{a_n}{n}\right|+$

$\left|\dfrac{a_{mn}}{mn}-\dfrac{a_m}{m}\right|<\dfrac{1}{n}+\dfrac{1}{m}$.

Problem 10.28(M.1195 ZK Proposed by ,Proposed by O.T.Izhboldin)

From

(1) $\quad a_n+a_m-\dfrac{1}{n+m}\leq a_{n+m}\leq a_n+a_m+\dfrac{1}{n+m}$ follows that for any $n\in\mathbb{N}$ holds inequalities

$a_n+a_1-\dfrac{1}{n+1}\leq a_{n+1} \iff a_1-\dfrac{1}{n+1}\leq a_{n+1}-\ a_n$ and

$a_{n+1}\leq a_n+a_1+\dfrac{1}{n+1} \iff a_{n+1}-a_n\leq a_1+\dfrac{1}{n+1}$

which implies

$a_{n+m}-\ a_m=\sum\limits_{k=m}^{n+m-1}(a_{k+1}-a_k)\geq\sum\limits_{k=m}^{n+m-1}\left(a_1-\dfrac{1}{k+1}\right) \iff$

(2) $\qquad na_1+a_m-\sum\limits_{k=1}^{n}\dfrac{1}{m+k}\leq a_{n+m}$ and

$a_{n+m}-\ a_m \leq \sum\limits_{k=m}^{n+m-1}\left(a_1+\dfrac{1}{k+1}\right)=na_1+\sum\limits_{k=1}^{n}\dfrac{1}{m+k} \iff$

(3) $\quad a_{n+m} \leq na_1+a_m+\sum\limits_{k=1}^{n}\dfrac{1}{m+k}$.

From **(3)** and right inequality of **(1)** we obtain

$a_n+a_m-\dfrac{1}{n+m} \leq a_{n+m} \leq na_1+a_m+\sum\limits_{k=1}^{n}\dfrac{1}{m+k} \implies$

$$a_n-na_1 \leq \frac{2}{m+n}+\sum_{k=1}^{n-1}\frac{1}{m+k}$$

and from **(2)** and left inequality of **(1)** we obtain

$na_1+a_m-\sum\limits_{k=1}^{n}\dfrac{1}{m+k} \leq a_{n+m} \leq a_n+a_m+\dfrac{1}{n+m} \implies$

$$-\left(\sum_{k=1}^{n-1}\frac{1}{m+k}+\frac{2}{m+n}\right) \leq a_n-na_1.$$

Thus, $|a_n-na_1| \leq \dfrac{2}{m+n}+\sum\limits_{k=1}^{n-1}\dfrac{1}{m+k}$ for any $n,m \in \mathbb{N}$ and since

$$|a_n-na_1| \leq \lim_{n\to\infty}\left(\frac{2}{m+n}+\sum_{k=1}^{n-1}\frac{1}{m+k}\right)=0$$

we finally obtain that $a_n-na_1=0$.

★Problem 10.28 (MRJ259)

Let $x_1,x_2,...,x_n$ be arbitrary increasing arithmetic progression $x_1,x_2,...,x_n$ such that $x_1^2+x_2^2+...+x_n^2=1$.

Since $x_k=x_1+(k-1)\,d, k=1,2,...,n$ then $x_1^2+x_2^2+...+x_n^2=1 \iff$

$x_1^2+(x_1+d)^2+(x_1+2d)^2+...+(x_1+(n-1)\,d)^2=1 \iff$

$nx_1^2+2x_1d\,(1+2+3+...+n-1)+d^2\left(1^2+2^2+...+(n-1)^2\right)=1 \iff$

$nx_1^2+x_1d\,(n-1)\,n+d^2\dfrac{(n-1)\,n\,(2n-1)}{6}=1 \iff$

$x_1^2+x_1d\,(n-1)=\dfrac{1}{n}-d^2\dfrac{(n-1)\,(2n-1)}{6} \iff$

$\left(x_1+\dfrac{d\,(n-1)}{2}\right)^2=\dfrac{1}{n}-d^2\dfrac{(n-1)\,(2n-1)}{6}+\dfrac{d^2\,(n-1)^2}{4}=$

$\dfrac{1}{n}-\dfrac{d^2\,(n-1)}{12}\,(4n-2-3(n-1)) \iff \left(x_1+\dfrac{d\,(n-1)}{2}\right)^2=$

$\dfrac{1}{n}-\dfrac{d^2\,(n^2-1)}{12} \implies \dfrac{1}{n}-\dfrac{d^2\,(n^2-1)}{12} \geq 0 \iff$

$d^2 \leq \dfrac{12}{n\,(n^2-1)} \iff d \leq \dfrac{2\sqrt{3}}{\sqrt{n\,(n^2-1)}}$ (since $d>0$).

Thus, we obtain upper bound for common difference d.

Let $d=d_*=\dfrac{2\sqrt{3}}{\sqrt{n\,(n^2-1)}}$ then quadratic equation

$nx_1^2+x_1d_*\,(n-1)\,n+d_*^2\dfrac{(n-1)\,n\,(2n-1)}{6}=1 \iff$

$\left(x_1 + \dfrac{d_* (n-1)}{2}\right)^2 = 0$

have only solution $x_1 = -\dfrac{d_* (n-1)}{2} = -2\sqrt{3}\sqrt{\dfrac{n-1}{n(n+1)}}$.

So, arithmetic progression $x_k = -2\sqrt{3}\sqrt{\dfrac{n-1}{n(n+1)}} + \dfrac{2\sqrt{3}(k-1)}{\sqrt{n(n^2-1)}}, k = 1, 2, ..., n$

satisfy $x_1^2 + x_2^2 + ... + x_n^2 = 1$ and maximize common difference d,

i.e. $\max d = \dfrac{2\sqrt{3}}{\sqrt{n(n^2-1)}}$.

Problem 10.29.(**Quickies-Q2(CRUX?)**)

Let $b_n := \left\lfloor \left(15+\sqrt{220}\right)^n + \left(15+\sqrt{220}\right)^{n+1} \right\rfloor = \left\lfloor \left(16+\sqrt{220}\right)\left(15+\sqrt{220}\right)^n \right\rfloor$ and $a_n := \left(16+\sqrt{220}\right)\left(15+\sqrt{220}\right)^n + \left(16-\sqrt{220}\right)\left(15-\sqrt{220}\right)^n$.

Then $a_0 = 32, a_1 = 920$ and a_n satisfy to the recurrence

(1) $\quad a_{n+1} - 30a_n + 5a_{n-1} = 0, n \in \mathbb{N}$.

Since $1 > \left(16-\sqrt{220}\right)\left(15-\sqrt{220}\right) \geq \left(16-\sqrt{220}\right)\left(15-\sqrt{220}\right)^n > 0$

and a_n is integer for $n \in \mathbb{N}$ and

$\left(16+\sqrt{220}\right)\left(15+\sqrt{220}\right)^n = a_n - 1 + 1 - \left(16-\sqrt{220}\right)\left(15-\sqrt{220}\right)^n$

we obtain $b_n = a_n - 1$.. By substitution $a_n = b_n + 1$ in the recurrence **(1)** we obtain recurrence for b_n :

(2) $\quad b_{n+1} - 30b_n + 5b_{n-1} = 24, n \in \mathbb{N}$ and $b_1 = 31, b_1 = 919$.

Let $r_n \equiv b_n \pmod{10}$ then for r_n we have recurrence

(3) $\quad r_{n+1} - 5r_{n-1} = 4$ and $r_0 = 1, r_1 = -1$.

Since $r_1 = -1$ and $r_{2k+1} = 5r_{2k-1} + 4, k \in \mathbb{N}$ we obtain $r_{2k-1} = -1$ for all $k \in \mathbb{N}$;

Since $r_2 = 9$ and $r_{2k+2} = 5r_{2k} + 4, k \in \mathbb{N}$ we obtain $r_{2k} \equiv 9 \pmod{10}$ for all $k \in \mathbb{N}$.

So, $b_n \equiv r_n \equiv 9 \pmod{10}$ for any $n \in \mathbb{N}$.